

Recognition of Generalized Network Matrices

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Abstract

In this thesis, we deal with binet matrices, an extension of network matrices. The main result of this thesis is the following. A rational matrix A of size $n \times m$ can be tested for being binet in time $O(n^6 m)$. If A is binet, our algorithm outputs a nonsingular matrix B and a matrix N such that $[B \ N]$ is the node-edge incidence matrix of a bidirected graph (of full row rank) and $A = B^{-1}N$.

Furthermore, we provide some results about Camion bases. For a matrix M of size $n \times m'$, we present a new characterization of Camion bases of M , whenever M is the node-edge incidence matrix of a connected digraph (with one row removed). Then, a general characterization of Camion bases as well as a recognition procedure which runs in $O(n^2 m')$ are given. An algorithm which finds a Camion basis is also presented. For totally unimodular matrices, it is proven to run in time $O((nm)^2)$ where $m = m' - n$.

The last result concerns specific network matrices. We give a characterization of nonnegative $\{\epsilon, \rho\}$ -noncorelated network matrices, where ϵ and ρ are two given row indexes. It also results a polynomial recognition algorithm for these matrices.

Keywords: network matrices, binet matrices and Camion bases.

Résumé

Dans cette thèse, nous nous penchons sur la notion de matrice binet qui étend celle de matrices réseau. Le résultat majeur de la thèse est le suivant: on peut tester si une matrice donnée A à coefficients rationnels et de taille $n \times m$ est binet en temps $O(n^6 m)$. Si A est binet, notre algorithme fournit une matrice inversible B et une matrice N telles que $[B \ N]$ est la matrice d'incidence sommet-arête d'un graphe biorienté de rang n et $A = B^{-1}N$.

En outre, nous exposons quelques résultats sur les bases de Camion. Pour une matrice M de taille $n \times m'$, nous présentons une nouvelle caractérisation des bases de Camion de M , lorsque M est la matrice d'incidence sommet-arête d'un graphe orienté connexe (auquel une ligne a été ôtée). Puis, une caractérisation générale des bases de Camion ainsi qu'une procédure de reconnaissance qui s'exécute en temps $O(n^2 m')$ sont données. Nous présentons également un algorithme permettant de trouver une base de Camion. Pour les matrices totalement unimodulaires, celui-ci nécessite un temps de calcul de l'ordre de $O((nm)^2)$ où $m = m' - n$.

Le dernier résultat intéressant concerne certaines matrices réseau spécifiques. Nous formulons une caractérisation des matrices réseau non négatives et non $\{\epsilon, \rho\}$ -corélées, où ϵ et ρ sont deux indices donnés de lignes. Il en résulte un algorithme polynomial de reconnaissance pour cette classe de matrices.

mots-clés: matrices réseau, matrices binet and bases de Camion.

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Chapter 1

Overview

Science sans conscience n'est que ruine de l'âme.
Rabelais (1485-1553).

1.1 Introduction

Network matrices play a central role in combinatorial optimization. They have been largely used for solving flow problems and integer programs. Their interest extends also to the field of matroids, graphs embeddable on the plane, etc...

There exist several polynomial-time algorithms to test if a given matrix is a network matrix. Such an algorithm was designed by Auslander and Trent [7], Gould [31], Tutte [55, 56, 57], Bixby and Cunningham [9] and Bixby and Wagner [10]. A famous one is Schrijver's method developed in [44] which adapts the matroidal ideas of Bixby and Cunningham [9] to matrices. The algorithm works by reducing the problem to a set of smaller problems, which can be handled easily. The smaller problems consist of deciding if a matrix with at most two non-zeros per column is a network matrix or not. The reduction is done by identifying rows of the matrix that correspond to cut-edges of the spanning tree, and then carrying on with the smaller matrices associated with the components. The time complexity of the algorithm developed in [9] is $O(n\alpha)$, where n and α are the number of rows and nonzero elements, respectively, in the input matrix.

Binet matrices, a generalization of network matrices, have been investigated by Kotnyek [36]. He provided an algorithm to determine the columns of a binet matrix using its graphical representation, and gave some geometrical properties of these matrices, by extending results about totally unimodular matrices (see Section 1.3). However, one question is left open: is it possible to recognize whether a given matrix is binet or not in time polynomial in its size?

In a parallel direction, a matroid called the signed-graphic matroid has been introduced by Zaslavsky at the beginning of the eighties [61]. Any binet matrix is a compact representation matrix of a signed-graphic matroid. Since then, this class of matroids continues to receive much attention in the mathematical literature, but several questions remain open, in particular the problem of determining whether a given matroid is signed-graphic or not. (See for example [1], [12], [16], [27], [41], [42], [43], [50], [51], [65] and [64].)

In this thesis, we turn to the problem of recognizing binet matrices. Our central result is the following.

Theorem 6.1 *A rational matrix of size $n \times m$ can be tested for being binet in time $O(n^6 m)$ using the algorithm Binet.*

Given a binet matrix A as input, the algorithm Binet outputs a nonsingular matrix B and a matrix N such that $[B \ N]$ is the node-edge incidence matrix of a bidirected graph (of full row rank) and $A = B^{-1}N$. Theorem 6.1 has different applications that are discussed in Section 1.3. Section 1.4 describes in some details the strategy employed for recognizing binet matrices and serves as a tool for reading the Chapters 6 to 11.

One key element of the algorithm Binet consists of finding a Camion basis of the matrix $[I \ A]$. We provide some new results about Camion bases. For a matrix M of size $n \times m'$, we present a characterization of Camion bases of M , whenever M is the node-edge incidence matrix of a connected digraph (with one row removed). Then, a general characterization of Camion bases as well as a recognition procedure which runs in $O(n^2 m')$ are given. There is no known polynomial-time algorithm to find a Camion basis in general. Fonlupt and Raco [24] described a finite procedure to find one based on the results of Camion, and gave an algorithm which runs in time $O(n^3 m^2)$ for totally unimodular matrices. We also present an algorithm which finds a Camion basis. For totally unimodular matrices, it is proven to run in time $O((nm)^2)$ where $m = m' - n$.

The last interesting result concerns specific network matrices. We give a characterization of nonnegative $\{\epsilon, \rho\}$ -noncorelated network matrices, where ϵ and ρ are two given row indexes. It results a polynomial recognition algorithm for this class of matrices.

1.2 Notions

We will present here the principal notions used in this thesis. More details are given in Chapters 2, 3 and 4. We assume familiarity of the reader with the elements of linear algebra, such as linear (in)dependence, rank, determinant, matrix, non-singular matrix, inverse, Gauss' algorithm for solving a system of linear equations, etc. A reader not familiar with the content of this section can consult, for example, Schrijver [44] and Welsh [58].

As always, \mathbb{Z} , \mathbb{Q} , and \mathbb{R} denote the set of integer, rational, and real numbers. \mathbb{N} contains the nonnegative integers. The *projective plane*, denoted by \mathbb{P}^2 , is the sphere (in \mathbb{R}^3) where any two antipodal points (x and $-x$) are identified.

Vectors and matrices whose elements are integers are called *integral*. That is, integral m -dimensional vectors are those in \mathbb{Z}^m , an integral matrix of size $n \times m$ is in $\mathbb{Z}^{n \times m}$. Similarly, *rational* vectors and matrices have elements from \mathbb{Q} . *Half-integral* vectors and matrices have elements that are integer multiples of $\frac{1}{2}$.

A matrix A is of *full row rank*, if its rank equals the number of its rows, or equivalently, if its row vectors are linearly independent. If A is of full row rank say n , a *basis* of A is a non-singular square submatrix of A of size n .

A set P of vectors in \mathbb{R}^m is a *polyhedron* if $P = \{x \mid Ax \leq b\}$ for an $n \times m$ matrix A and n -dimensional vector b . The vectors of a polyhedron are called its *points*. An *extreme point* or *vertex* of a polyhedron $P = \{x \mid Ax \leq b\}$ is a point determined by m linearly independent equations from $Ax = b$. Every extreme point of P can arise as an optimal solution of $\max\{c^T x \mid x \in P\}$ for a suitably chosen c . For any $1 \leq i \leq n$, let us denote by $A_{i\bullet}$ the i th

row of A and let $P^{(i)}$ be the polyhedron obtained from P by removing the i th row from the system $Ax \leq b$. If $P^{(i)} \neq P$, then $P \cap \{A_{i\bullet}x = b_i\}$ is called a *facet* of P .

If P has at least one vertex, then P is called *integral*, if all of its vertices are integral. An integral polyhedron P provides integral optimal solutions for $\max\{c^T x \mid x \in P\}$ for any c . Similarly, if P is half-integral, then the optimal solutions are half-integral.

A set S of vectors is *convex* if it satisfies: if $x, y \in S$ and $0 \leq \lambda \leq 1$, then $\lambda x + (1 - \lambda)y \in S$. The *convex hull* of a set S of vectors is the smallest convex set containing S , and is denoted $\text{conv}(S)$.

Let $M \in \mathbf{R}^{n \times m'}$ be a matrix of rank n , $\mathcal{H}(M) = \{\{x \in \mathbf{R}^n : c^T x = 0\} : c \text{ is a column of } M\}$ and B a basis of M . $\mathcal{H}(M)$ splits up \mathbf{R}^n into a set S of full dimensional cones (regions). A *simplex region* is one which has exactly n facets. The matrix B is called a *Camion basis* if the corresponding hyperplanes determine the facets of a simplex region in S . It is known that there always exists a Camion basis. After some column permutations, we may write $M = [B \ N]$. It is possible to show that B is a Camion basis if and only if by multiplying some rows and columns of the matrix $B^{-1}N$, the resulting matrix is nonnegative. Geometrically, $B^{-1}N \geq 0$ means that the column vectors of N are contained in the cone generated by B .

An (*undirected*) *graph* is a pair $G = (V, E)$, where V is a finite set, and E is a family of unordered pairs of elements of V . A *directed graph* or *digraph* is a pair $G = (V, E)$, where V is a finite set, and E is a finite family of ordered pairs of elements of V . The elements of V are called the *vertices* or *nodes* of G , and the elements of E are called the *edges* of G .

The *node-edge incidence matrix* of a graph G has its rows and columns associated with the nodes and edges of the digraph. The non-zeros in a column associated with edge e stand in the rows that correspond to the endnodes of e . If G is directed, then heads get positive signs and tails get negative signs, otherwise all non-zero entries are equal to 1. We denote by IMD the node-edge incidence matrix of a digraph. See Figure 1.1. The rank of the node-edge incidence matrix In of a connected directed graph G on n nodes is $n - 1$. Moreover, by deleting any row, In can be made a full row rank matrix In' . The bases of In' correspond to spanning trees of G . An *RIMD*, or restricted IMD, is an IMD with (linearly) redundant rows removed.

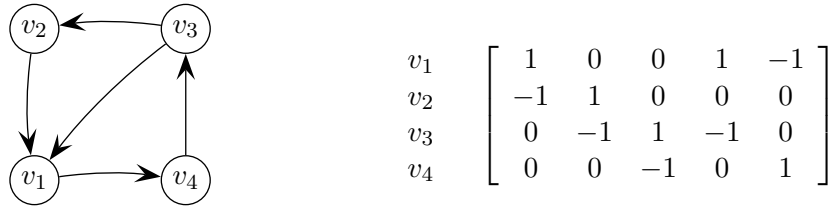


Figure 1.1: A digraph G (on the left) and the node edge incidence matrix In of G .

Let $G = (V, E)$ be a digraph and In the $V \times E$ -incidence matrix of G . Let In' be an RIMD obtained from In and B a basis of In' such that $In' = [B \ N]$ (up to column permutations). The matrix $A = B^{-1}N$ is called a *network matrix*.

By assuming that G is connected, the basis B corresponds to a spanning tree of G (as mentioned above) whose subgraphs are called *basic*. The rows and columns of the network matrix are associated with the tree and non-tree edges, respectively. For any non-tree edge f , we find the unique cycle (called the fundamental cycle) which contains f and some edges

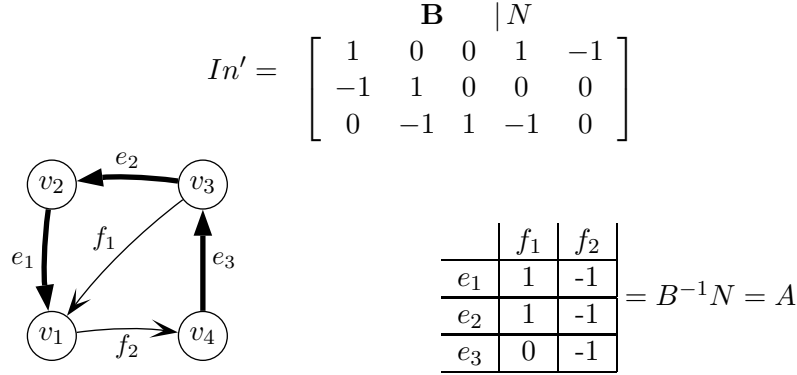


Figure 1.2: An RIMD In' (above) obtained from the IMD In described in Figure 1.1 by removing the last row. For the basis B consisting of the three first columns of In' , the corresponding network matrix and network representation are given at the bottom. The edges e_1, e_2 and e_3 corresponding to the basis B form a spanning tree of G . The edges f_1 and f_2 correspond to the first and second column of N , respectively.

from the tree. The column of the network matrix corresponding to f will contain ± 1 in the rows of the tree edges in its fundamental cycle and 0 elsewhere. The signs of the non-zeros depend on the directions of the edges. If walking through the tree along the fundamental cycle starting at the tail of f , a tree edge lies in the same direction, it gets a positive sign, if it lies in the opposite direction, it gets a negative sign. See Figure 1.2.

Let A be a nonnegative connected network matrix, and suppose that we are given two row indexes, say ϵ and ρ ($\epsilon \neq \rho$). Let $G(A)$ be a network representation of A and q a basic path in $G(A)$ containing e_ϵ and e_ρ . If q passes through one of the edges e_ϵ and e_ρ forwardly and through the other backwardly, then we say that e_ϵ and e_ρ are *alternating* in $G(A)$, otherwise *nonalternating*. If A has a network representation in which e_ϵ and e_ρ are alternating and another one in which they are nonalternating, then N is said to be $\{\epsilon, \rho\}$ -*noncorelated*, otherwise $\{\epsilon, \rho\}$ -*corelated*.

A matrix A is *totally unimodular* if each subdeterminant of A is 0, +1, or -1. In particular, each entry in a totally unimodular matrix is 0, +1, or -1. It is known that any network matrix is totally unimodular.

A bidirected graph $G = (V, E)$ on node set $V = \{v_1, \dots, v_n\}$ and with edge set $E = \{e_1, \dots, e_m\}$ may have four kinds of edges. A *link* is an edge with two distinct endnodes; a *loop* has two identical endnodes; a *half-edge* has one endnode, and a *loose edge* has no endnode at all.

Every edge is signed with + or - at its endnodes. That is, links or loops can be signed with +-, ++ or --; half-edges have only one sign, + or -. If an edge is signed with + at an endnode, then this node is an *in-node* or *head* of the edge. An endnode signed with - is called the *out-node* or *tail* of the edge. One can think of the value + as indicating that the edge is directed into the node, - indicating direction away from the node. All edges, except directed edges, are called *bidirected edges*. See Figure 1.3.

Every edge e is given a *sign*, denoted by $\sigma_e \in \{+, -\}$. The signing convention we adopt is that the sign of an edge is negative if and only if it is bidirected.

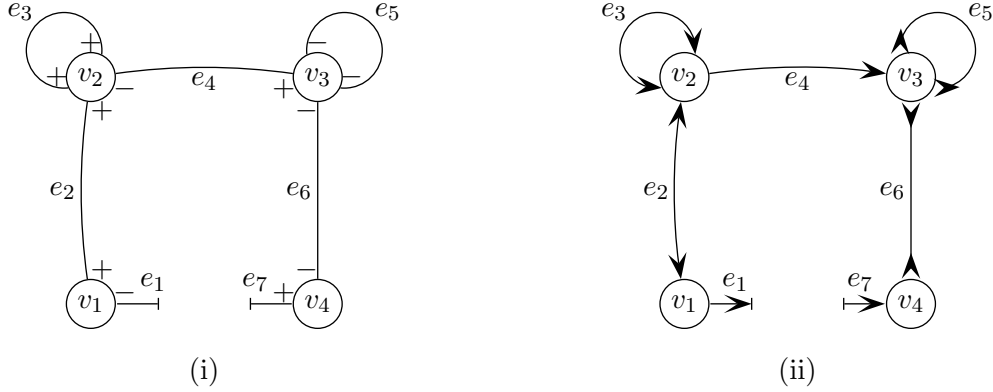


Figure 1.3: Possible graphical representations of a bidirected graph

Let us define the *(node-edge) incidence matrix* $In(G)$ of a bidirected graph G called an IMB. The rows and columns of $In(G)$ are identified with the nodes and edges of G , respectively. An entry (i, j) of $In(G)$ is 1 (resp., -1) if e_j is a link or a half-edge entering (resp., leaving) v_i , 2 (resp., -2) if e_j is a negative loop entering (resp., leaving) v_i , 0 otherwise. An *RIMB*, or restricted IMB, is an IMB with (linearly) redundant rows removed. As an example, the node-edge incidence matrix of the bidirected graph depicted in Figure 1.3 is:

$$In = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

A matrix A is called a *binet matrix* if there exist a full row rank incidence matrix In of a bidirected graph G and a basis B of it such that $In = [B \ N]$ (up to column permutations) and $A = B^{-1}N$. As any IMD is an IMB, the class of binet matrices contains all network matrices.

A *matroid* M is a finite ground set S and a collection ϕ of subsets of S such that (I1)-(I3) are satisfied.

(I1) $\emptyset \in \phi$.

(I2) If $X \in \phi$ and $Y \subseteq X$ then $Y \in \phi$.

(I3) If X, Y are members of ϕ with $|X| = |Y| + 1$ there exists $x \in X \setminus Y$ such that $Y \cup \{x\} \in \phi$.

Subsets of S in ϕ are called the *independent sets*, a maximal independent subset in S is a *basis* of M . A *circuit* is a minimal dependent subset of S . If B is a basis of M and $s \in S \setminus B$, then there is a unique circuit in $B \cup \{s\}$, called the *fundamental circuit of s with respect to B* . There are several different but equivalent ways to define a matroid. For example, one can define a matroid on a given ground set through its bases, rank-function, circuits, or fundamental circuits.

A standard example of matroids is when S is a finite set of vectors from a vector space over a field \mathbb{F} and ϕ contains the linearly independent subsets of S . This kind of matroid is called the *linear matroid*. If for a matroid M there exists a field \mathbb{F} such that M is a linear matroid over \mathbb{F} , then M is *representable* over \mathbb{F} . The matrix In made up of the vectors in S is called a *standard representation matrix* of M . There is a one-to-one correspondence between linearly independent columns of In and independent sets in M , so the linear matroid $M = M(In)$ can be fully given by its representation matrix In .

There is another, more compact representation matrix of a linear matroid $M(In)$. To get it, first delete linearly dependent (over \mathbb{F}) rows from In , if there are any, then choose a basis B of M . It corresponds to a basis of In , also denoted by B , and let N be the remaining matrix in In . Then the matrix $A = B^{-1}N$ (where the inverse of B is computed over the field \mathbb{F}) is called a *compact representation matrix* of M over \mathbb{F} .

1.3 Applications

In combinatorial optimization and geometry, many of the most frequently used algorithms exploit the discrete structure and properties of the matrices involved in the problems: network simplex method, integer programs with totally unimodular constraint matrix, Tardos' method for linear programming,... Our purpose here is a description of four applications resulting from a recognition procedure for binet matrices. Binet matrices have strong connections with some linear and integer programs that are easily solvable, and a class of polyhedrons with half-integral vertices. Moreover, they are related to relatively unknown matroids and a characterization of the graphs embeddable on the projective plane.

Consider the linear program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & 0 \leq x \leq \alpha \end{aligned} \tag{1.1}$$

where $c, \alpha \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times m}$ for some $n, m \in \mathbb{N}$, and some entries of α may be infinite. If the constraint matrix in (1.1) is an RIMD, then the problem is called a transshipment problem or more simply a network problem in the literature, and the interpretation of (1.1) is the following. Variables correspond to flows in edges, while the constraints of the system " $Ax = b$ " correspond to supply (if $b_r < 0$), demand (if $b_r > 0$) or flow conservation (if $b_r = 0$) requirements at the vertices; the inequalities $0 \leq x \leq \alpha$ are capacity constraints. The objective is to find a minimum cost flow subject to these constraints. The optimum can be found by using the *network simplex method*, a successful cross between the algebra of the simplex method and the combinatorics of the flow algorithms.

If A is an RIMB, then (1.1) is a generalization of the network problem and is called a *bidirected LP*. The optimal solution of a bidirected LP can be achieved by general-purpose methods, like the simplex algorithm, or the strongly polynomial algorithm of Tardos [52]. Kotnyek [36] proved that a bidirected LP can be solved using a generalized network simplex method which is not polynomial in the worst case, but for most of the practical problems it is much more efficient than the simplex method, or the strongly polynomial method of Tardos (see the reference notes in [3]).

Now assume that we are to solve (1.1) with the additional constraint that x is integral. Given any undirected graph $G = (V, E)$, if A is the node-edge incidence matrix of G and $\alpha_i = 1$ for all $1 \leq i \leq m$, then (1.1) is known as the optimum b -matching problem: given a real numerical weight c_e for each edge $e \in E$ and an integer b_v for each node $v \in V$, find in G , if there is one, a subgraph G' which has degrees b_v at nodes v and whose edges have maximum weight-sum. Whenever A is an RIMB (and x is required to be integral), the problem (1.1) is called a *bidirected IP* and generalizes the b -matching problem. Edmonds proved that bidirected IPs are solvable in polynomial time (see [20, 21, 37]).

It is known that the solution set of the system " $Ax = b$ " is unchanged under elementary row operations, like multiplying a row by a nonzero constant or adding a row to another one. Using Gauss' algorithm, we may assume that the matrix A in (1.1) is in standard form (i.e., the first n columns of A form the identity matrix). Then a natural question is whether one can convert (1.1) to a bidirected LP or IP (when x is required to be integral) using elementary row operations on the system " $Ax = b$ ". We show in Section 4.4 that this is equivalent to the problem of recognizing binet matrices as stated in the next theorem.

Theorem 4.13 *Let A be a real matrix in standard form. Then A can be converted to an RIMB using elementary row operations if and only if A is binet.*

Our procedure for recognizing binet matrices provides a way of transforming any rational matrix A into an RIMB using elementary row operations, whenever A is binet.

The second application deals with polyhedral geometry. Given a rational matrix A of size $n \times m$ and a rational vector b of size n , consider $P = \{x : x \geq 0, Ax \leq b\}$. A primary problem of integer programming is to find the *integer hull* $P_I = \text{conv}\{P \cap \mathbb{Z}^m\}$ of the polyhedron P . Provided that A is integral, it is known that $P = P_I$ for all integral right hand side vectors b , if and only if A is totally unimodular. In terms of integer programming, totally unimodular matrices are the integral matrices for which $\max\{c^T x \mid Ax \leq b, x \geq 0\}$ has integral optimal solutions for any c and any integral b .

There are situations, however, that take us beyond total unimodularity. Do we know the non-integral matrices A that ensure integral optimal solutions of $\max\{c^T x \mid Ax \leq b, x \geq 0\}$ for any c and any integral b ? Or what can we say about matrices A that ensure integral optimal solutions for only a special set of right hand side b ? These questions are not independent. If A is rational, then one can find a nonnegative integer k , such that if we multiply every row of A by k , we get an integral matrix, kA . But then instead of inequalities $Ax \leq b$, we have $kAx \leq kb$ and we deal with polyhedra that are required to be integral for only special b' vectors, namely for those whose elements are integer multiples of k . For example, if $k = 2$, so the elements of A are halves of integers, then we are to characterize integral matrices A' for which $\{x \mid A'x \leq b', x \geq 0\}$ is integral for all even vectors b' . Or equivalently, we examine matrices that provide half-integral vertices for polyhedra with integral right hand sides.

A matrix is called *k -regular* ($k \in \mathbb{N}$) if for each of its non-singular square submatrices π , $k\pi^{-1}$ is integral. k -regularity is the property that takes over the role of total unimodularity in the theory of rational matrices that ensure integral vertices for polyhedra with special right hand sides.

Theorem 4.14 (Kotnyek [36]) *A rational matrix A is k -regular, if and only if the poly-*

hedron $\{x \mid Ax \leq kb, x \geq 0\}$ is integral for any integral vector b .

Kotnyek [36] proved that every binet matrix is 2-regular. Binet matrices seem to form a very big subclass of 2-regular matrices. It is even conjectured by Appa that a characterization theorem should exist for a decomposition of 2-regular matrices into binet matrices, the transpose of binet matrices and probably other non-binet matrices. If such a decomposition theorem would exist, then a procedure for recognizing binet matrices would be an essential tool for recognizing 2-regular matrices.

Now we describe an interpretation of binet matrices in the field of matroids. For any bidirected graph G , the linear matroid of the node-edge incidence matrix of G is called signed-graphic. Thus any binet matrix based on the bidirected graph G is the compact representation matrix (over \mathbb{R}) of a signed-graphic matroid. Furthermore, given a binet matrix A , by multiplying the columns of A with nonempty $\pm\frac{1}{2}$ -support by -2 , we shall show that the resulting matrix is a compact representation matrix of a signed-graphic matroid over $GF(3)$ (the field of cardinality 3).

Finally, we state a result discussed in Section 4.6.

Theorem 1.1 *Let G be a 2-connected graph and T a spanning tree. Let \vec{G} be a digraph obtained from G by orienting the edges, and A the network matrix with respect to \vec{G} and $\vec{T} \subseteq \vec{G}$. Then A^T is a binet matrix if and only if G is embeddable on \mathbb{P}^2 .*

Then an algorithm for recognizing binet matrices yields a way of testing if a given graph is embeddable on \mathbb{P}^2 . However, in the literature, there exist several methods for doing this that are much more efficient than the one deriving from our recognition algorithm for binet matrices (see [38]).

1.4 Recognizing binet matrices

In this section, we give more technical details of the binet recognition algorithm. Before that, we provide different definitions and notations. This section may help for reading Chapters 6 to 11.

Let A be a rational matrix with row set R and column set S . $(A)_{ij}$ or A_{ij} or a_{ij} denotes the element in row $i \in R$ and column $j \in S$. For $R' \subseteq R$ and $S' \subseteq S$, $A_{R' \bullet}$ (respectively, $A_{\bullet S'}$) denotes the set of rows of A indexed by R' (respectively, columns of A indexed by S').

For any $j \in S$, denote by $s(A_{\bullet j}) = \{i : A_{ij} \neq 0\}$ the support of $A_{\bullet j}$ and by $s_k(A_{\bullet j}) = \{i : A_{ij} = k\}$ the k -support, for any $k \in \mathbb{R}$. A k -entry of A is an entry of A equal to k . For a set $I \subseteq \mathbb{R}$, an I -matrix is a matrix all of whose entries are in I . For $R' \subseteq R$, $f(R') = \{j : s(A_{\bullet j}) \cap R' \neq \emptyset\}$ and $\chi_{R'}^R$ denotes the characteristic vector or incidence vector of the subset R' of R , given by $(\chi_{R'}^R)_i = \begin{cases} 1 & \text{if } i \in R' \\ 0 & \text{Otherwise} \end{cases}$ for all $i \in R$. $A^{\frac{1}{2} \rightarrow 1}$ denotes the matrix obtained from A by replacing each $\frac{1}{2}$ -entry by 1.

A walk in a bidirected graph is a sequence $(v_1, e_1, v_2, \dots, v_{t-1}, e_{t-1}, v_t)$ where v_i and v_{i+1} are endnodes of edge e_i ($i = 1, \dots, t-1$), including the case where $v_i = v_{i+1}$ and e_i is a half-edge. If the walk consists of only links and it does not cross itself, i.e $v_i \neq v_j$ for $1 < i < t$,

$1 \leq j \leq t$, $i \neq j$, then it is a *path*. A closed walk which does not cross itself (except at $v_1 = v_t$) and goes through each edge at most once is called a *cycle*. So a loop, a half-edge or a closed path can make up a cycle. In Figure 1.3, there are exactly four cycles. The *sign of a cycle* is the product of the signs of its edges, so we have a *positive cycle* if the number of negative edges (or bidirected edges) in the cycle is even; otherwise, the cycle is a *negative cycle*. Obviously, a negative loop or a half-edge always makes a negative cycle. A *full cycle* in a bidirected graph is a cycle different from a half-edge.

A bidirected graph is *connected*, if there is a path between any two nodes. A *tree* is a connected bidirected graph which does not contain a cycle. A connected bidirected graph containing exactly one cycle is called a *1-tree*, indicative of the fact that a 1-tree consists of a tree and one additional edge. If the unique cycle in a 1-tree is negative, then we will call it a *negative 1-tree*.

Let G be a bidirected graph and $In(G)$ its node-edge incidence matrix. Any submatrix Q of the incidence matrix $In(G)$ can be obtained by row and column deletions. By analogous operations, a bidirected graph $G(Q)$ can be obtained from G (see Section 3.2 for more details).

Now let B be a basis of $In(G)$, N such that $In(G) = [B \ N]$ and $A = B^{-1}N$. Edges in the subgraph $G(B)$ of G are called *basic* edges. The edges of G that are not in $G(B)$ (i.e., those of $G(N)$) are the *nonbasic* edges. By Lemma 3.4, the graph $G(B)$ consists of negative 1-tree components. The unique cycle in a basic component is called a *basic cycle*. Basic edges, respectively nonbasic ones, are in one-to-one correspondance with rows of A , respectively columns of A .

The bidirected graph G with the set of basic and nonbasic edges marked is called a *binet representation* of A (not unique in general) and is denoted as $G(A)$. By removing the nonbasic edges from $G(A)$, we obtain a subgraph of G called a *basic binet representation* of A . There is a similar definition for network matrices. Let $m = m' - n$. Basic edges of the binet representation will be denoted by e_1, e_2, \dots, e_n and the nonbasic ones f_1, f_2, \dots, f_m , so that e_i ($1 \leq i \leq n$), respectively f_j ($1 \leq j \leq m$), corresponds to the i th row, respectively j th column of A . For any nonbasic edge f_j , the subgraph of $G(A)$ with edge set $\{f_j\} \cup \{e_i : i \in s(A_{\bullet, j})\}$ (and without any isolated node) is called the *fundamental circuit* of f_j . One can prove that the fundamental circuit of any nonbasic edge f_j falls in one of the following categories (see Figure 1.4 for an example).

- (i) it is a loose edge (equal to f_j), or
- (ii) a positive cycle, or
- (iii) a pair of negative cycles with exactly one common node, or
- (iv) a pair of disjoint negative cycles along with a minimal connecting path.

A binet representation of A is called *proper* if each basic component has exactly one bidirected edge (contained in the basic cycle), this one is entering, and there is another edge in the basic cycle entering one endnode of the basic bidirected edge; these two edges and the in-node incident with them are said to be *central*.

A $\frac{1}{2}$ -binet representation is a proper binet representation in which every basic cycle is a half-edge. A matrix is said to be $\frac{1}{2}$ -binet if it has a $\frac{1}{2}$ -binet representation.

A *cyclic representation* of a matrix is a proper binet representation of the matrix having exactly one basic cycle, and this one is full. For R a row index set, an *R -cyclic representation*

	f_1	f_2	f_3	f_4	f_5	f_6	f_7
e_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	0
e_2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	0	0
e_3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	-1	0
e_4	1	1	1	1	2	0	0
e_5	1	1	0	0	1	0	0
e_6	0	0	1	0	1	0	0
e_7	0	0	1	0	0	0	1
e_8	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$

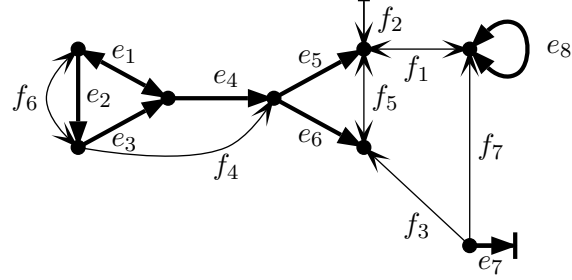


Figure 1.4: An example of a binet matrix with a binet representation of it.

of a matrix A denotes a cyclic representation such that R is the edge index set of the basic cycle, and this cycle is contained in the fundamental circuit of at least one nonbasic edge f_j (or equivalently $R \subseteq s(A_{\bullet j})$ for at least one column index j of A). A *bicyclic representation* of a matrix is a proper binet representation of the matrix having exactly two basic cycles, and these are full. At last, we say that a binet representation is an $\{\epsilon, \rho\}$ -central representation, if it is cyclic, e_ϵ and e_ρ are edges of the basic cycle incident with one common central node, and one of the edges e_ϵ and e_ρ is bidirected. The matrix A is said to be *cyclic* (respectively, *bicyclic*,...) if and only if it has a cyclic (respectively, bicyclic,...) representation.

Let A be a given rational matrix of size $n \times m$. Our purpose now is to determine whether A is binet or not. To achieve this, we first transform A into a nonnegative matrix A' such that A' is binet if and only if A is binet, or outputs that A is not binet (see Chapter 5). We prove that even in the worst case we need at most $O((nm)^2)$ operations to get A' . We will show that any entry of a binet matrix is in $\{0, \pm 1, \pm 2, \pm \frac{1}{2}\}$. Therefore, we also check whether each entry of A' belongs to $\{0, 1, 2, \frac{1}{2}\}$, otherwise A is not binet.

Now we may assume that A is a matrix with 0-, 1-, 2- or $\frac{1}{2}$ -entries. In the second step of the algorithm, we carry out a procedure called Decomposition described in Chapter 6. This procedure is performed on the matrix A and the empty row index subset as input. The matrix A is called $\frac{1}{2}$ -equisupported if for any column indexes j and j' such that $s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset$ and $s_{\frac{1}{2}}(A_{\bullet j'}) \neq \emptyset$, we have $s_{\frac{1}{2}}(A_{\bullet j}) = s_{\frac{1}{2}}(A_{\bullet j'})$. We mention here a very important theorem.

Theorem 6.2 *The matrix A is binet if and only if one of the following three statements is valid:*

- 1) A is bicyclic and $\frac{1}{2}$ -equisupported, or
- 2) A is cyclic and without any $\frac{1}{2}$ -entry, or
- 3) the procedure Decomposition with input A and row index subset $Q = \emptyset$ provides a binet representation of A .

By Theorem 6.2, if the procedure Decomposition does not provide any binet representation of A , then we distinguish two cases: either A has a $\frac{1}{2}$ -entry and we test if A is $\frac{1}{2}$ -equisupported and bicyclic, or A has no $\frac{1}{2}$ -entry and we determine whether A has a cyclic representation. If

we fail, we conclude that A is not binet. Since any cyclic matrix has either an $\{i\}$ -cyclic representation for some row index i , or an $\{\epsilon, \rho\}$ -central representation for some row indexes ϵ and ρ , recognizing whether A is cyclic can be reduced to the recognition of R^* -cyclic ($|R^*| = 1$) and R^* -central matrices ($|R^*| = 2$).

The procedure Decomposition takes advantage of a simple property of binet matrices: if A has a binet representation $G(A)$ and the columns $A_{\bullet j}$ and $A_{\bullet j'}$ have non-empty non-equal intersecting $\frac{1}{2}$ -support for some $1 \leq j, j' \leq m$, then $s_{\frac{1}{2}}(A_{\bullet j}) \cap s_{\frac{1}{2}}(A_{\bullet j'})$ is the edge index set of a basic cycle in $G(A)$. Provided that A has at least one $\frac{1}{2}$ -entry, we locate different row index subsets, say R_1, \dots, R_δ , and related disjoint submatrices A_1, \dots, A_δ in A . We also compute some matrix τ and another one $A(\tau)$ obtained from A by removing all rows and columns intersecting any submatrix A_i ($1 \leq i \leq \delta$) and adding τ .

We shall prove that whenever A has a binet representation $G(A)$, the subgraph $G(A_i)$ is an R_i -cyclic representation of A_i , for $i = 1, \dots, \delta$; moreover, a $\frac{1}{2}$ -binet representation of $A(\tau)$ can be obtained from $G(A)$ as follows: delete in $G(A)$ all edges of the subgraphs $G(A_1), \dots, G(A_\delta)$ and the remaining isolated nodes, then add a basic half-edge at each left node of any $G(A_i)$ for $1 \leq i \leq \delta$. Using an R_i -cyclic representation of A_i , for $i = 1, \dots, \delta$, and a $\frac{1}{2}$ -binet representation $G(A(\tau))$ of $A(\tau)$, one can compute a binet representation of A . Thus the procedure Decomposition uses a subroutine for recognizing nonnegative R^* -cyclic matrices and another one for recognizing nonnegative $\frac{1}{2}$ -binet matrices.

Let us describe some main ideas for the recognition of nonnegative R^* -cyclic matrices. See Chapters 7 and 8 for more details. Suppose that we are given the matrix A and a row index subset R^* of A . Let $S^* = \{j : s(A_{\bullet j}) \cap R^* \neq \emptyset\}$. We decompose the matrix $A_{\overline{R^*} \times \overline{S^*}}$ into a maximum number of blocks and denote by E_1, \dots, E_b the row index sets of the different blocks. Each set E_l is called a *bonsai*. For all $1 \leq l \leq b$, we compute row index subsets of E_l , called E_l -paths, denoted by $E_l^1, \dots, E_l^{m(l)}$, as well as a *bonsai* matrix N_l . The matrix $A_{E_l \times \overline{S^*}}$ (without zero columns) and the vectors $\chi_{E_l^k}^{E_l}$ are submatrices of N_l . Then we define a digraph denoted by D and a matrix $O(R^*)$ containing the submatrix $A_{R^* \bullet}$. The graphical interpretation of these objects is the following.

Suppose that A has an R^* -cyclic representation $G(A)$. Let $1 \leq l \leq b$. The bonsai E_l corresponds to the edge index set of a basic subtree in $G(A)$ (outside the basic cycle) called a *bonsai* and denoted by B_l . For any $j \in S^*$, the intersection of the fundamental circuit of the nonbasic edge f_j with some bonsai B_l is either empty, or a directed path called a B_l -path, or a union of two directed paths, called B_l -paths, starting at a same node. Each E_l -path is the edge index set of a B_l -path (there is a one-to-one mapping between B_l -paths and E_l -paths). The bonsai B_l is a basic network representation of the matrix $(N_l)_{E_l \bullet}$. The digraph D contains some information about the feasible connections between the bonsais B_l ($1 \leq l \leq b$). From $G(A)$ one can derive a basic network representation of $O(R^*)$ as follows: delete all nonbasic edges, contract some bonsais, switch at some nodes and "cut" the basic cycle at some node so that the basic cycle becomes a path, and add some directed edge at each endnode of this path. By computing some feasible spanning forest of D as well as a network representation of each bonsai matrix N_l ($1 \leq l \leq b$) and of the matrix $O(R^*)$, if they exist, one can construct an R^* -cyclic representation of A .

The subroutine of the procedure Decomposition for recognizing nonnegative $\frac{1}{2}$ -binet matrices is presented in Chapter 9. The described method is based on the following fact: if A has a $\frac{1}{2}$ -binet representation $G(A)$ such that S is the index set of all nonbasic bidirected edges (and these are entering), then $A' = \begin{bmatrix} A \\ (\chi_S^{\{1, \dots, m\}})^T \end{bmatrix}$ has a $\{n+1\}$ -cyclic representation. So our purpose is to compute a column index set S such that whenever A is binet S corresponds to the index set of nonbasic bidirected edges in some $\frac{1}{2}$ -binet representation of A . Let $S_2 = \{j : s_2(A_{\bullet j}) \neq \emptyset\}$. We compute a family \mathcal{S} of pairwise disjoint row index subsets of A and a column index subset $S(\mathcal{S})$ satisfying the following: whenever A has a $\frac{1}{2}$ -binet representation $G(A)$, each element of \mathcal{S} corresponds to the edge index set of a basic (negative) 1-tree in $G(A)$, and the matrix $A' = \begin{bmatrix} A \\ (\chi_{S(\mathcal{S}) \cup S_2}^{\{1, \dots, m\}})^T \end{bmatrix}$ is $\{n+1\}$ -cyclic. Moreover, given a $\{n+1\}$ -cyclic representation of A' , it is easy to deduce a $\frac{1}{2}$ -binet representation of A (by contracting the basic loop with index $n+1$). Thus, the recognition of nonnegative $\frac{1}{2}$ -binet matrices is reduced to the recognition of nonnegative R^* -cyclic matrices with $|R^*| = 1$.

Now we turn to the problem of recognizing nonnegative bicyclic matrices. This problem is deeply studied in Chapter 10. Suppose that we are given the matrix A , this one is $\frac{1}{2}$ -equisupported and has at least one $\frac{1}{2}$ -entry. Let $S_{\frac{1}{2}} = \{j : s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset\}$ and $R^* = s_{\frac{1}{2}}(A_{\bullet j})$ for any $j \in S_{\frac{1}{2}}$. We decompose the matrix $A_{\bullet \overline{S_{\frac{1}{2}}}}$ into blocks whose row index sets are denoted by C_1, \dots, C_r called *cells*. Provided that A is bicyclic, one can prove that R^* is the edge index set of both basic cycles in any bicyclic representation of A ; the difficulty consists in finding a partition of R^* into subsets say R_1 and R_2 ($R^* = R_1 \uplus R_2$) such that A has a bicyclic representation where R_1 and R_2 are the edge index sets of the basic cycles. Actually, we will partition the set \mathcal{K} of cells intersecting R^* .

Suppose that A has a bicyclic representation $G(A)$. One can prove that any cell C_k ($1 \leq k \leq r$) is the edge index set of a subtree in $G(A)$. We say that $G(A)$ *induces the bipartition* $\Sigma(\mathcal{K}) = \{\mathcal{K}_I, \mathcal{K}_{II}\}$, where \mathcal{K}_I (respectively, \mathcal{K}_{II}) is the set of cells corresponding to subtrees in one basic maximal 1-tree of $G(A)$ (respectively, the other 1-tree). We will prove that if $A_{C_k \bullet}^{\frac{1}{2} \rightarrow 1}$ is not a network matrix for some $C_k \in \mathcal{K}$, then the corresponding subtree in $G(A)$ contains a central edge. Similarly, when $s_{\frac{1}{2}}(A_{\bullet j}) \not\subseteq R^*$ for some $j \in S_{\frac{1}{2}}$.

These observations restrict the choices of feasible bipartitions and motivate the definition of *bicompatible* bipartition in \mathcal{K} . Then, we define an equivalence relation over the set of bicompatible bipartitions in \mathcal{K} . We prove that the cardinality of the resulting quotient set \mathcal{S} is bounded by a constant, actually 18. Moreover, we show the following: provided that A is bicyclic, if a bicompatible bipartition $\Sigma(\mathcal{K})$ is induced by some bicyclic representation of A , then for any bicompatible bipartition $\Sigma'(\mathcal{K})$ equivalent to $\Sigma(\mathcal{K})$, there exists a bicyclic representation of A inducing $\Sigma'(\mathcal{K})$.

At last, for any representant $\Sigma(\mathcal{K}) = \{\mathcal{K}_I, \mathcal{K}_{II}\}$ of an equivalence class in \mathcal{S} , we denote by $M_i(\Sigma)$ the matrix $A_{\cup_{C_k \in \mathcal{K}_i} C_k \bullet}$ without zero columns and $R_i^*(\Sigma) = \cup_{C_k \in \mathcal{K}_i} C_k \cap R^*$ for $i = I$ and II ; we also define a matrix $N(\Sigma)$ containing every block of $A_{\overline{R^*} \times S_{\frac{1}{2}}}$ whose row index set does not intersect R^* . We will prove that whenever A has a bicyclic representation $G(A)$ inducing a bipartition $\Sigma(\mathcal{K})$, the subgraph $G(M_i(\Sigma))$ is an $R_i^*(\Sigma)$ -cyclic representation of $M_i(\Sigma)$ for $i = I$ and II . Furthermore, if we consider the union of subtrees with edge index

set C_k ($C_k \notin \mathcal{K}$) and all fundamental circuits of nonbasic edges with index in $S_{\frac{1}{2}}$, then by deleting the nonbasic edges and replacing each basic cycle by a basic half-edge, we obtain a basic $\frac{1}{2}$ -binet representation of $N(\Sigma)$ (with exactly two basic half-edges). For any bicompatible bipartition $\Sigma(\mathcal{K})$, by computing an $R_i^*(\Sigma)$ -cyclic representation of $M_i(\Sigma)$ for $i = I$ and II and some $\frac{1}{2}$ -binet representation of $N(\Sigma)$, if they exist, we derive a bicyclic representation of A .

Finally, we describe the way of recognizing nonnegative R^* -central matrices ($|R^*| = 2$) without any $\frac{1}{2}$ -entry. See Chapter 11 for more details. Let A be a $\{0, 1, 2\}$ -matrix and ϵ and ρ two row indexes. Our method for determining whether A is $\{\epsilon, \rho\}$ -central can be viewed as a generalization of Schrijver's method [44] for recognizing network matrices. Let $R^* = \{\epsilon, \rho\}$ and $S^* = \{j : s(A_{\bullet j}) \cap R^* \neq \emptyset\}$. We decompose the matrix $A_{\overline{R^*} \times \overline{S^*}}$ into (a maximum number of) blocks whose row index sets are denoted by E_1, \dots, E_b and called *bonsais*, and compute a digraph D in the same way as for the recognition of R^* -cyclic matrices (see Chapter 7).

Our first task is to analyze the significative kinds of bonsais. Let $S_0 = \{j \in S^* : \epsilon, \rho \in s(A_{\bullet j})\}$, $S_1 = \{j \in S^* : \epsilon \in s(A_{\bullet j}), \rho \notin s(A_{\bullet j})\}$ and $S_2 = \{j \in S^* : \epsilon \notin s(A_{\bullet j}), \rho \in s(A_{\bullet j})\}$. We say that a bonsai E_l is *shared* if there exist $j_1 \in S_1$ and $j_2 \in S_2$ such that $E_l \cap s(A_{\bullet j_1}) \neq \emptyset$ and $E_l \cap s(A_{\bullet j_2}) \neq \emptyset$.

Suppose that A has an $\{\epsilon, \rho\}$ -central representation $G(A)$. Each bonsai E_l corresponds to the edge index set of a subtree denoted by B_l in $G(A)$ and called a *bonsai*. A bonsai B_l ($1 \leq l \leq b$) is said to be shared if and only if E_l is shared. Let T be the basic maximal 1-tree in $G(A)$. A basic subgraph of $G(A)$ is *on the right of* $\{e_\epsilon, e_\rho\}$ if it is contained in the basic connected subtree of $T \setminus \{e_\epsilon, e_\rho\}$ containing the central node incident to e_ϵ and e_ρ , otherwise *on the left of* $\{e_\epsilon, e_\rho\}$. On the other hand, it may happen that two shared bonsais are in the situation described in Figure 1.5. (It would be preferable in Figure 1.5 that $v_{l'} = v_{l,1}$ or $v_{l'} = v_{l,2}$.)

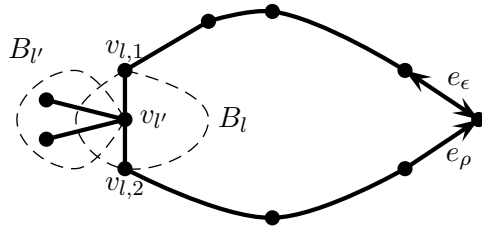


Figure 1.5: An illustration of two (shared) bonsais B_l and $B_{l'}$ in some basic $\{\epsilon, \rho\}$ -central representation of A . The bonsais B_l and $B_{l'}$ have a common node $v_{l'}$; the intersection of B_l with the basic cycle is a path between two nodes say $v_{l,1}$ and $v_{l,2}$, $v_{l'} \neq v_{l,1}$ and $v_{l'} \neq v_{l,2}$.

In order to bypass this situation, an initialization procedure is performed on A , and outputs a matrix A' such that A is $\{\epsilon, \rho\}$ -central if and only if A' is $\{1, \rho\}$ -central, and A' satisfies a certain condition called the *assumption* \mathcal{A} that prevents the situation described in Figure 1.5. Then two cases are distinguished: either $S_0 \neq \emptyset$ or $S_0 = \emptyset$.

Assume first that $S_0 = \emptyset$. Suppose that A has an $\{\epsilon, \rho\}$ -central representation $G(A)$. The main observation is the following: if we look at the succession of bonsais intersecting the edge

set of the basic cycle in $G(A)$ (see Figure 1.6), we may identify a first sequence of non-shared bonsais, a second of shared bonsais and a third of non-shared bonsais. In the sequence of shared bonsais (if not empty), the first and last bonsais, say B_l and $B_{l'}$, are of particular interest: the pair $(E_l, E_{l'})$ or the singleton (E_l) (in case $E_l = E_{l'}$) is called a *left-extreme* set. If there is no shared bonsai intersecting the edge set of the basic cycle, then for any shared bonsai B_l on the left of $\{e_\epsilon, e_\rho\}$, the set (E_l) is also called *left-extreme*. For instance, in Figure 1.6 a (respectively, b), the pair (E_3, E_5) (respectively, (E_3) or (E_4)) is left-extreme.

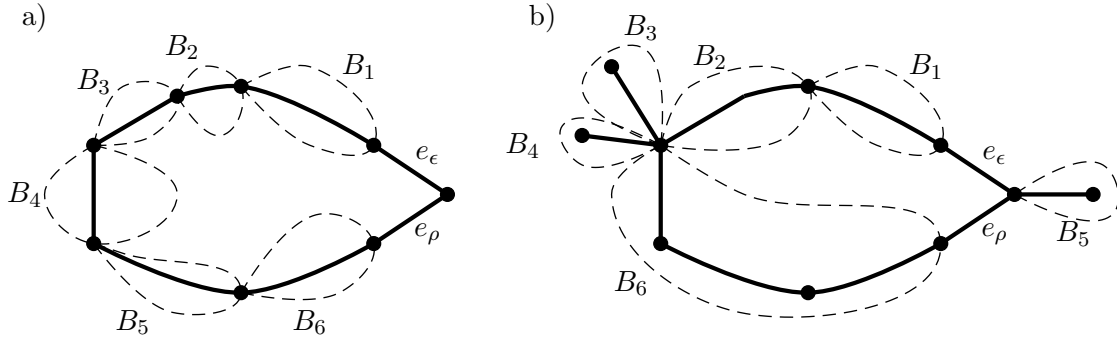


Figure 1.6: An illustration of the bonsais intersecting the basic cycle in some basic $\{\epsilon, \rho\}$ -central representation of a binet matrix A , where B_3 , B_4 and B_5 are assumed to be shared. The sequence of shared bonsais intersecting the edge set of the basic cycle is B_3 , B_4 and B_5 in case a, and is empty in case b. In case b however, B_3 and B_4 have exactly one common node with the basic cycle.

Now consider any shared bonsai E_l . By assuming that there exists an $\{\epsilon, \rho\}$ -central representation $G(A)$ of A such that B_l is on the left of $\{e_\epsilon, e_\rho\}$, is it possible to compute the indexes of edges belonging to the basic cycle and B_l ? Surprisingly it is, provided that the left-extreme set is known and of cardinality 2. This tremendous fact is at the core of our procedure.

The notion of left-extreme set has motivated the definition of left-compatible set (depending only on the matrix A) as we will see. Provided that A has an $\{\epsilon, \rho\}$ -central representation $G(A)$, a left-extreme set of bonsais is necessarily left-compatible. For any left-compatible set U ($|U| \leq 2$), we define a U -spanning pair (j_1, j_2) of column indexes as follows: $j_1 \in S_1$, $j_2 \in S_2$; if $U = (E_u)$, then $s(A_{\bullet j_i}) \cap E_u \neq \emptyset$ for $i = 1$ and 2 , and if $U = (E_u, E_{u'})$, then $s(A_{\bullet j_1}) \cap E_{u'} \neq \emptyset$ and $s(A_{\bullet j_2}) \cap E_u \neq \emptyset$. We also define $V(j_1, j_2) = \{E_l : s(A_{\bullet j_i}) \cap E_l \neq \emptyset \text{ for } i = 1 \text{ or } 2, \text{ or } E_l \text{ is shared}\}$. We will prove that whenever A has an $\{\epsilon, \rho\}$ -cyclic representation $G(A)$ and U is left-extreme, the union of the bonsais B_l (with $E_l \in V(j_1, j_2)$) and $\{e_\epsilon, e_\rho\}$ yields a basic 1-tree in $G(A)$. We also define an instance $\Omega(U, j_1, j_2)$ of 2-SAT, where each variable x_l is associated to a bonsai $E_l \in V(j_1, j_2)$. One necessary condition for a variable to receive a value 0 or 1 is that some matrix containing the matrix $A_{E_l \cup \{\epsilon, \rho\} \bullet}$ or $A_{E_l \bullet}$ is a network matrix.

Given a truth assignment of $\Omega(U, j_1, j_2)$ and a network representation of every bonsai matrix N_l with $E_l \notin V(j_1, j_2)$, we can construct an $\{\epsilon, \rho\}$ -central representation $G(A)$ of A such that for any $E_l \in V(j_1, j_2)$, we have $x_l = 0 \Leftrightarrow B_l$ is on the left of $\{e_\epsilon, e_\rho\}$ in $G(A)$, and for any $E_u \in U$ $x_u = 0$.

If we consider a left-compatible set U with only one element say E_u , we may have to determine whether the matrix $A_{E_u \cup \{\epsilon, \rho\} \bullet}$ has a network representation such that e_ϵ and e_ρ are nonalternating. Thus it is necessary to develop a procedure for recognizing nonnegative $\{\epsilon, \rho\}$ -noncorelated network matrices. This is the purpose of Section 11.4.

To deal with the case $S_0 \neq \emptyset$, the strategy is very similar to the case $S_0 = \emptyset$. We define a notion of S_0 -straight bonsai. A necessary condition for a bonsai E_l to be S_0 -straight is that $s(A_{\bullet j}) \cap E_l = s(A_{\bullet j'}) \cap E_l \neq \emptyset$ for any $j, j' \in S_0$.

Suppose that A has an $\{\epsilon, \rho\}$ -central representation $G(A)$ and $S_0 \neq \emptyset$. We prove that if a bonsai B_l intersects the edge set of the basic cycle then E_l is S_0 -straight. However, it might happen that a bonsai B_l is on the right of $\{e_\epsilon, e_\rho\}$ and E_l is S_0 -straight. A bonsai E_u is called *right-extreme* if E_u is S_0 -straight, B_u is on the right of $\{e_\epsilon, e_\rho\}$ and there is no S_0 -straight bonsai in $G(A)$ "closer" than B_u to the basic cycle. The set of right-extreme bonsais is proved to be of cardinality at most two. This set has motivated the definition of *right-compatible* set.

Let $V'_0 = \{E_l : s(A_{\bullet j}) \cap E_l \neq \emptyset \text{ for some } j \in S_0 \text{ or } E_l \text{ is shared}\}$. For any right-compatible set U , we define an instance $\Lambda(U)$ of 2-SAT, where each variable x_l is associated to a bonsai $E_l \in V'_0$. Given a truth assignment of $\Lambda(U)$ and a network representation of every bonsai matrix N_l with $E_l \notin V'_0$, we can construct an $\{\epsilon, \rho\}$ -central representation $G(A)$ of A such that for any $E_l \in V'_0$, we have $x_l = 0 \Leftrightarrow B_l$ is on the left of $\{e_\epsilon, e_\rho\}$ in $G(A)$.

1.5 Contents

Chapter 1 resumes the thesis. In Section 1.1 we introduce the main results of the thesis, and in Section 1.2 the necessary terms. Some applications are presented in Section 1.3. Section 1.4 provides some technical details of the most significative part of the thesis: a binet recognition algorithm. This section, Section 1.5, gives the content of each chapter and section.

In Chapter 2, we introduce the main notations and notions. Section 2.1 is about vectors, matrices and polyhedrons, Section 2.2 about undirected and directed graphs. Section 2.3 contains the definition of network matrices and totally unimodular matrices with some known related results.

In Chapter 3 we define bidirected graphs as a common generalization of undirected and directed graphs. Basic definitions about bidirected graphs are in Section 3.1. In Section 3.2 we define the node-edge incidence matrix of a bidirected graph and characterize linear independence and dependence in this matrix. We also present some operations on bidirected graphs and the corresponding operations in their node-edge incidence matrix.

We embark on generalizing network matrices in Chapter 4, where we define binet matrices. We describe the graphical interpretation of binet matrices in Section 4.1 and some operations on binet matrices in Section 4.2. Section 4.3 introduces some particular binet representations. In Section 4.4 we discuss linear and integer programming with the node-edge incidence matrix of a bidirected graph or a binet matrix as constraint matrix. Section 4.5 is about the class of 2-regular matrices which contains all binet matrices. Section 4.6 deals with matroids. To exhibit the connection of binet matrices to existing special classes of matroids, we introduce signed graphs and then the signed-graphic matroid based on these graphs. Some elementary notions about matroids are given in Subsection 4.6.2.

Chapter 5 is about Camion bases. We present some characterizations of Camion bases in Sections 5.1 and 5.2 and an algorithm which finds a Camion basis in Section 5.3. This

Chapter can be read independently of the other ones.

In Chapter 6, we turn to the problem of recognizing binet matrices by using the results of Chapters 7 to 11. Section 6.1 describes the general framework of an algorithm called Binet which determines whether a given matrix is binet in time polynomial in its size. Section 6.2 describes the second main step of the algorithm Binet, that is a subroutine called Decomposition.

In Chapter 7, we develop a structure for the recognition of R^* -cyclic and R^* -central matrices. Section 7.1 serves as a tool for the next section. In Section 7.2, we compute some matrices called bonsai matrices. In Section 7.3, we define a digraph denoted by D , and in Section 7.4, a feasible spanning forest of D . Section 7.5 describes an algorithm for computing a feasible spanning forest of D , if one exists.

Given a rational matrix A and a row index subset R^* of A , one determines whether A has an R^* -cyclic representation in Chapter 8. An informal sketch of the recognition procedure is given in Section 8.1, and the procedure itself, called RCyclic, with the formal proof of its correctness appears in Section 8.2.

Similarly, Chapters 9 and 10 are about the recognition of $\frac{1}{2}$ -binet and bicyclic matrices, respectively.

In Chapter 11, given a rational matrix A and two row indexes, say ϵ and ρ , we turn to the problem of recognizing whether A has an $\{\epsilon, \rho\}$ -central representation. By letting $S_0 = \{j : \epsilon, \rho \in s(A_{\bullet j})\}$, a recognition procedure is presented in Section 11.2 for the case $S_0 = \emptyset$, and in Section 11.3 for the case $S_0 \neq \emptyset$. Section 11.1 provides some definitions and an initialization procedure for the main procedures of Sections 11.2 and 11.3. In Section 11.4, we give a characterization of nonnegative $\{\epsilon, \rho\}$ -noncorelated matrices and a polynomial recognition procedure for these matrices.

In Chapter 12, we analyze the pertinence of our procedure for recognizing binet matrices and present possible alternatives and future developments.

Chapter 2

Basic Notions and Definitions

Here we define the notions used throughout this thesis. The notations we used are standard. We assume familiarity of the reader with the elements of linear algebra, such as linear (in)dependence, rank, determinant, matrix, non-singular matrix, inverse, Gauss' algorithm for solving a system of linear equations, etc. A reader not familiar with the content of this section can consult, for example, Schrijver [44] or Diestel [17].

As always, \mathbb{Z} , \mathbb{Q} , and \mathbb{R} denote the set of integer, rational, and real numbers. \mathbb{N} contains the nonnegative integers. The set of positive real and integer numbers are \mathbb{R}_+ and \mathbb{N}_+ , respectively. The greatest integer smaller than $x \in \mathbb{R}$ is denoted by $\lfloor x \rfloor$. The *projective plane*, denoted by \mathbb{P}^2 , is the sphere (in \mathbb{R}^3) where any two antipodal points (x and $-x$) are identified.

For a set S , $|S|$ denotes the cardinality of S . Two sets S and S' are called *disjoint*, if $S \cap S' = \emptyset$. If S and S' are two disjoint sets, then $S \uplus S'$ denotes the disjoint union of S and S' .

A *partial order* is a binary relation " \leq " over a set S which is reflexive, antisymmetric, and transitive, i.e., for all a, b and c in S , we have:

- $a \leq a$ (reflexivity);
- if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry);
- if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

A *partially ordered set* (or *poset*) is a set equipped with a partial order relation.

We write $f(x) = O(g(x))$ for real-valued functions f and g , if there exists a constant C such that $f(x) \leq Cg(x)$ for all x in the domain.

If we consider an optimization problem like

$$\min\{\phi(x) \mid x \in S\}$$

where S is a set and $\phi : S \rightarrow \mathbb{R}$, then any element x of S is called a *feasible solution* for the minimization problem. A feasible solution attaining the maximum is called an *optimum* (or *optimal solution*).

2.1 Vectors, matrices and polyhedrons

If $x = [x_1, \dots, x_n]$ and $y = [y_1, \dots, y_n]$ are row vectors, we write $x \leq y$ if $x_i \leq y_i$ for $i = 1, \dots, n$. Similarly for column vectors. If A is a matrix of size $n \times m$, and b is a column vector of size n , then the matrix A is called the *constraint matrix* of the system of linear equations " $Ax = b$ ".

Vectors and matrices whose elements are integers are called *integral*. That is, integral m -dimensional vectors are those in \mathbb{Z}^m , an integral matrix of size $m \times n$ is in $\mathbb{Z}^{m \times n}$. Similarly, *rational* vectors and matrices have elements from \mathbb{Q} . *Half-integral* vectors and matrices have elements that are integer multiples of $\frac{1}{2}$. For a set R and $R' \subseteq R$, we denote by $\chi_{R'}^R$ the *characteristic vector* or *incidence vector* of the subset R' of R , given by $(\chi_{R'}^R)_i = \begin{cases} 1 & \text{if } i \in R' \\ 0 & \text{Otherwise} \end{cases}$ for all $i \in R$.

Let A be a rational matrix with row set R and column set S . $(A)_{ij}$ or A_{ij} or a_{ij} denotes the element in row $i \in R$ and column $j \in S$. For $R' \subseteq R$ and $S' \subseteq S$, $A_{R' \times S'}$ denotes the submatrix of A whose rows are indexed by R' and columns by S' . $A_{R' \bullet}$ (respectively, $A_{\bullet S'}$) denotes the set of rows of A indexed by R' (respectively, columns of A indexed by S'). A_{R^2} is equal to $A_{R \times R}$. For $S' \subseteq S$, $\overline{S'} = S \setminus S'$. $O_{n \times m}$ is a zero $n \times m$ matrix for some integers n and m . Sometimes, I denotes a square matrix with ones in the diagonal and zeros elsewhere. We denote by kA the matrix obtained from A by multiplying all elements by k ($k \in \mathbb{R}$). Thus A is a half-integral matrix, if and only if $2A$ is integral. A^T is the transpose of A .

For any $j \in S$, denote by $s(A_{\bullet j}) = \{i : A_{ij} \neq 0\}$ the support of $A_{\bullet j}$ and by $s_k(A_{\bullet j}) = \{i : A_{ij} = k\}$ the k -support, for any $k \in \mathbb{R}$. A k -entry of A is an entry of A equal to k . For a set $I \subseteq \mathbb{R}$, an I -matrix is a matrix all of whose entries are in I . For $R' \subseteq R$, we denote $f(R') = \{j : s(A_{\bullet j}) \cap R' \neq \emptyset\}$. $A^{\frac{1}{2} \rightarrow 1}$ denotes the matrix obtained from A by replacing each $\frac{1}{2}$ -entry by 1. A is called $\frac{1}{2}$ -equisupported if for any column indexes j and j' such that $s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset$ and $s_{\frac{1}{2}}(A_{\bullet j'}) \neq \emptyset$, we have $s_{\frac{1}{2}}(A_{\bullet j}) = s_{\frac{1}{2}}(A_{\bullet j'})$.

If $A = \left[\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array} \right]$ is a matrix that has submatrices A_1 and A_2 in the upper left and bottom right corners, respectively, and zeros outside them, then this structure of A is its *decomposition* to components A_1 and A_2 ; the matrices A_1 and A_2 are called *blocks* of A . A matrix A is said to be *connected* if and only if A has no zero row or column and A can not be decomposed into two blocks using row and column permutations.

The determinant of a square matrix A is denoted $\det(A)$. The *rank* of a matrix A is denoted $\text{rank}(A)$. A matrix is of *full row rank*, if its rank equals the number of its rows, or equivalently, if its row vectors are linearly independent. A *basis* of a full row rank A is a non-singular square submatrix of A of size $\text{rank}(A)$. The rank of the node-edge incidence matrix A of a connected directed graph G on n nodes is $n - 1$. Moreover, by deleting any row, A can be made a full row rank matrix A' . The bases of A' correspond to spanning trees of G .

A set S of vectors is *convex* if it satisfies: if $x, y \in S$ and $0 \leq \lambda \leq 1$, then $\lambda x + (1 - \lambda)y \in S$. The *convex hull* of a set S of vectors is the smallest convex set containing S , and is denoted $\text{conv}(S)$.

A matrix whose entries are zeros and ones, is said to be an *interval* matrix if its rows can be permuted in such a way that the 1's in each column occur consecutively.

Let A and A' be $n \times m$ matrices. We say that A is *projectively equivalent* to A' if there

is a nonsingular $n \times n$ matrix π such that $A' = \pi A$.

An operation on a matrix A that will be frequently used is *pivoting*, that is, the following transformation:

$$A = \begin{bmatrix} \alpha & \mathbf{c} \\ \mathbf{b} & D \end{bmatrix} \longrightarrow A' = \begin{bmatrix} \frac{1}{\alpha} & \frac{\mathbf{c}}{\alpha} \\ -\frac{\mathbf{b}}{\alpha} & D - \frac{\mathbf{bc}}{\alpha} \end{bmatrix},$$

where α is a non-zero entry, \mathbf{b} a column vector, \mathbf{c} a row vector, and D a submatrix of A .

A set P of vectors in \mathbb{R}^m is a *polyhedron* if $P = \{x \mid Ax \leq b\}$ for an $n \times m$ matrix A and n -dimensional vector b . The vectors of a polyhedron are called its *points*. An *extreme point* or *vertex* of a polyhedron $P = \{x \mid Ax \leq b\}$ is a point determined by m linearly independent equations from $Ax = b$. Every extreme point of P can arise as an optimal solution of $\max\{cx \mid x \in P\}$ for a suitably chosen c .

If P has at least one vertex (in which case P is called *pointed*), then P is called *integral*, if all of its vertices are integral. An integral polyhedron P provides integral optimal solutions for $\max\{cx \mid x \in P\}$ for any c . Similarly, if P is half-integral, then the optimal solutions are half-integral.

2.2 Graphs and digraphs

An (*undirected*) *graph* is a pair $G = (V, E)$, where V is a finite set, and E is a family of unordered pairs of elements of V . The elements of V are called the *vertices* or *nodes* of G , and the elements of E are called the *edges* of G . For $v, w \in V$, an edge in E connecting v and w is denoted (v, w) . The term "family" in the definition of graph means that a pair of vertices may occur several times in E .

A *directed graph* or *digraph* is a pair $G = (V, E)$, where V is a finite set, and E is a finite family of ordered pairs of elements of V . The elements of V are called the *vertices* or *nodes* of G , and the elements of E are called the *edges* or *arcs* of G . For $v, w \in V$, an arc in E from v to w is denoted (v, w) . The vertices v and w are called the *tail* and the *head* of the arc (v, w) , respectively. So the difference with undirected graphs is that orientations are given to the pairs.

For an (undirected or directed) graph $G = (V, E)$, a pair occurring more than once in E is called a *multiple edge*. Graphs without multiple edges are called *simple*. For a graph G , $V(G)$ denotes the vertex set of G and $E(G)$ the edge set. Two vertices v and w are *adjacent* if they are contained in a same edge. The edge (v, w) is said to be *incident with* the vertex v and with the vertex w , and conversely. The vertices v and w are called the *endnodes* of the edge (v, w) . The number of edges incident with a vertex v is called the *degree* of v .

For $G = (V, E)$ and $G' = (V', E')$ two graphs, we set $G \cup G' := (V \cup V', E \cup E')$ and $G \cap G' := (V \cap V', E \cap E')$. A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If E' is the family of all edges of G which have both endnodes in V' , then G' is said to be *induced by* V' . For a graph G and a subgraph $G' \subseteq G$, the *edge incidence vector* of G' is the vector in $\mathbb{R}^{|E(G)|}$, denoted by $\chi_{G'}$, satisfying

$$(\chi_{G'})_e = \begin{cases} 1 & \text{if } e \in E(G'), \\ 0 & \text{otherwise.} \end{cases}$$

A simple graph is *complete* if E is the set of all pairs of vertices. Two graphs $G = (V, E)$ and $G' = (V', E')$ are called *isomorphic*, denoted $G \simeq G'$, if there is a bijection $f : V \rightarrow V'$ satisfying, for all $v, w \in V$,

$$(v, w) \in E \Leftrightarrow (f(v), f(w)) \in E'.$$

A graph $G = (V, E)$ is called *bipartite* if V can be partitioned into two classes V_1 and V_2 such that each edge of G contains a vertex in V_1 and a vertex in V_2 . The sets V_1 and V_2 are called *colour classes*.

Let $G = (V, E)$ be an undirected or directed graph. A *walk* is a sequence $(v_1, e_1, v_2, \dots, v_{t-1}, e_{t-1}, v_t)$ where v_i and v_{i+1} are endnodes of edge e_i ($i = 1, \dots, t-1$). The node v_1 (resp., v_t) is called the *initial node* (resp., *terminal node* or *endnode*) of the walk. If the walk does not cross itself, i.e. $v_i \neq v_j$ for $1 < i < t$, $1 \leq j \leq t$, $i \neq j$, then it is a *path*. If $v_1 = v_t$, the walk is called *closed*. A closed path is called a *cycle*. In all these subgraphs the directions of the edges are not relevant. When they are, we speak about *directed paths*, *directed cycle*, etc.. The *length* of a walk is its number of edges.

Removing an edge e from G or *deleting an edge* means removing it and we write $G - \{e\} = (V, E - \{e\})$. *Removing a node* from G or *deleting a node* means removing it and all edges incident with it. We write $G - \{v\}$ the graph obtained from G by deleting v . For an edge $e = (x, y)$, we denote by G/e the graph obtained from G by *contracting* the edge e into a new vertex v_e , which becomes adjacent to all the former neighbours of x and y (if G is directed, we keep the same orientation of the edges). When we call a graph *minimal* or *maximal* with some property, we are referring to the subgraph relation.

The graph G is *connected* if G is not empty and there is a path between any two nodes. A *tree* is a connected graph which does not contain a cycle. A *subtree* is a connected subgraph of a tree. A subgraph $G' = (V', E')$ of $G = (V, E)$ is a *spanning tree* of G if $V' = V$ and G' is a tree. A *star* is a tree with at most one vertex of degree larger than 1.

A maximal connected subgraph of G is called a *component* of G . A vertex v is called a *cutvertex* if $G - \{v\}$ has one more component than G . An edge is a *cut-edge*, if it separates two parts of the graph, i.e., after deleting a cut-edge the graph has one more connected component. G is called 2-connected if $|G| > 2$ and $G - \{v\}$ is connected for any vertex $v \in G$.

Sometimes, it is convenient to consider one vertex of a tree as special; such a vertex is then called the *root* of this tree. A tree with a fixed root is called a *rooted tree*. If a rooted tree consists of a set of directed paths entering (respectively, leaving) the root, then it is called an *in-rooted tree* (respectively, *out-rooted tree*).

Suppose that $G = (V, E)$ is a digraph. G is called *strongly connected* if between any two nodes v and w there exists a directed path from v to w . A set $V' \subseteq V$ is said to be *closed in* G if for any arc (v, v') in G with $v \in V'$, we have that $v' \in V'$. A vertex v in G is called a *sink*, if $\{v\}$ is closed in G . If G has no directed cycle, then the *height* of a vertex $v \in V$ is the length of the shortest directed path in G from v to a sink vertex, and the set of sink vertices is denoted by $Sink(G)$; moreover, if there is a directed path from a node v to a node w , then w is called a *descendant* of v , and v an *ancestor* of w .

The *node-edge incidence matrix* of a graph G has its rows and columns associated with the nodes and edges of the digraph. The non-zeros in a column associated with edge e stand in the rows that correspond to the endnodes of e . If G is directed, then heads get positive signs and tails get negative signs, otherwise all non-zero entries are equal to 1. We denote by

IMD the node-edge incidence of a digraph. An *RIMD*, or restricted *IMD*, is an *IMD* with (linearly) redundant rows removed.

An undirected graph G is said to be *embeddable on a surface S* (for instance $S = \mathbb{R}$) if it is isomorphic to a graph (or pair) $G' = (V', E')$ with the following properties:

- (i) $V', E' \subseteq S$;
- (ii) different vertices are different points on S ;
- (iii) every edge is a curve, that is a continuous injective mapping from the interval $[0, 1]$ to S , between its endnodes;
- (iv) the interior of an edge contains no vertex and no point of any other edge.

2.3 Network matrices and totally unimodular matrices

In this Section, we introduce the very well-known class of network matrices and totally unimodular matrices. As an example, we see that interval matrices are particular network matrices and we show that network matrices constitute a subclass of totally unimodular matrices. Moreover, we describe Hoffman's and Kruskal's characterization of totally unimodular matrices, providing the link between total unimodularity and integer linear programming. Then, we present the transshipment problem and the related question of deciding whether a matrix is a network matrix with an application to integer programming. We also discuss a deep theorem stating that each totally unimodular matrix arises by certain compositions from network matrices and from certain 5×5 matrices. Finally, we provide some definitions that are used in Chapters 7 and 11.

Let $G = (V, E)$ be a digraph and In the $V \times E$ -incidence matrix of G . Let In' be an *RIMD* obtained from In , B a basis of In' and suppose that $In' = (B \ N)$. The matrix $A = B^{-1}N$ is called a *network matrix*.

Edges in G corresponding to columns of B (resp., N) are called *basic* (resp., *nonbasic*) edges. Basic edges, respectively nonbasic ones, are in one-to-one correspondance with rows of A , respectively columns of A . The digraph G with the indication of basic and nonbasic edges is called a *network representation* of A (not unique in general) and is denoted $G(A)$.

Suppose now that G is connected. It is not difficult to see that the basis B of In' corresponds to a spanning tree $T = (V, E_0)$ of G . It is known that for $e \in E_0$ and an edge $f = (u, v) \in E$:

$$a_{e,f} := \begin{cases} 1 & \text{if the unique u-v-path in } T \text{ passes through } e \text{ forwardly,} \\ -1 & \text{if the unique u-v-path in } T \text{ passes through } e \text{ backwardly,} \\ 0 & \text{if the unique u-v-path in } T \text{ does not pass through } e. \end{cases} \quad (2.1)$$

The unique closed path going from u to v through f and then from v to u in T is called the *fundamental cycle* of f .

Interval matrices are special network matrices, in which the spanning tree is a directed path. The non-tree edges then can be associated with subpaths. By definition, the non-zero elements in each column of an interval matrix are equal to 1, and there is a permutation of the rows such that the resulting matrix has consecutive ones in each column. Thus the columns

can be considered as characteristic vectors of intervals on a line with finite segments. Hence the name interval matrix.

A matrix A is *totally unimodular* if each subdeterminant of A is 0, +1, or -1. In particular, each entry in a totally unimodular matrix is 0, +1, or -1. A matrix of full row rank is *unimodular* if A is integral, and each basis of A has determinant ± 1 . It is easy to see that a matrix A is totally unimodular if and only if the matrix $[I \ A]$ is unimodular.

We prove here that every network matrix is totally unimodular.

Theorem 2.1 *Network matrices are totally unimodular.*

Proof. Let A be a network matrix. By definition, there exist an RIMD In and a basis B of In such that $In = [B \ N]$ and $A = B^{-1}N$. Let S be a square submatrix of A with nonzero determinant. Then $\det(S) = \det(S')$ for some basis S' of $[I \ A]$. Since the matrix $[B \ In]$ is an IMD, it is not difficult to prove that any basis of $[B \ In]$ has a determinant equal to ± 1 . This implies that $\det(B) = \pm 1$ and $\det(BS') = \pm 1$ (because BS' is a basis of $[B \ In]$). Thus $\det(S) = \pm 1$. ■

The link between unimodularity or total unimodularity and integer linear programming is given by the following fundamental results (see [44] for the proof).

Theorem 2.2 *Let A be an integral matrix of full row rank. Then A is unimodular if and only if for each integral vector b the polyhedron $\{x \mid x \geq 0; Ax = b\}$ is integral.*

Theorem 2.3 ((Hoffman and Kruskal's theorem)) *Let A be an integral matrix. Then A is totally unimodular if and only if for each integral vector b the polyhedron $\{x \mid x \geq 0; Ax \leq b\}$ is integral.*

Consider the linear programm

$$\begin{aligned} \min \quad & c^T x \\ \text{s.c.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{2.2}$$

where $c \in \mathbb{R}^m$, $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times m}$ for some $n, m \in \mathbb{N}$. Suppose that the constraint matrix A is integral and of full row rank. Then by Theorem 2.3, A is unimodular if and only if for any integral right-hand side b , there exists an integral optimal solution.

If the constraint matrix in (2.2) is an RIMD, then the problem is called a transshipment problem or more simply a network problem in the literature and the interpretation of (2.2) is the following. Variables correspond to flows in edges, while the constraints correspond to supply (if $b_r < 0$), demand (if $b_r > 0$) or flow conservation (if $b_r = 0$) requirements at the vertices; the objective is to find a minimum cost flow subject to these constraints.

Very efficient solution techniques are available for (2.2) such as the Edmonds-Karp scaling method [23], and the network simplex method [14, 15, 40]). Second, if b is integral, then any of the standard solution techniques will find an integral solution.

Suppose that the constraint matrix A in (2.2) is not an RIMD and we wish to convert the problem (2.2) to a transshipment problem in order to take advantage of the integrality properties of transshipment problems. It would be nice to have an algorithm for deciding when

a given matrix can be converted to an RIMD by using only elementary row operations. This is equivalent to the problem of recognizing network matrices as stated in the next theorem (see [9]).

Theorem 2.4 *Let A be a real $n \times m$ matrix in standard form (the first n columns of A form the identity matrix). Then A is projectively equivalent to an RIMD if and only if A is a network matrix.*

Proof. We have $A = [I \ A']$ where A' is a submatrix of A . Suppose that there exists a nonsingular matrix π such that $\pi[I \ A'] = In$ and In is an RIMD. It suffices to prove that A' is a network matrix. Denote $N = \pi A'$. So π is a basis of the RIMD $In = [\pi N]$ and $A' = \pi^{-1}N$.

Conversely, if A is a network matrix, then $A' = B^{-1}N$ where $[B \ N]$ is an RIMD and B a basis of $[B \ N]$, then BA is an RIMD. \blacksquare

There exist several algorithms to test if a given matrix is a network matrix. Such an algorithm was designed by Auslander and Trent [7], Gould [31], Tutte [55, 56, 57], Bixby and Cunningham [9] and Bixby and Wagner [10]. A famous one is Schrijver's method developed in [44] which adapts the matroidal ideas of Bixby and Cunningham [9] to matrices. The algorithm works by reducing the problem to a set of smaller problems, which can be handled easily. The smaller problems consist of deciding if a matrix with at most two non-zeros per column is a network matrix or not. The reduction is done by identifying rows of the matrix that correspond to cut-edges of the spanning tree, and then carrying on with the smaller matrices associated with the components. We mention here a theorem stated in [9] that will be used several times in the following chapters.

Theorem 2.5 *Let A be a real matrix of size $n \times m$ and α be the number of nonzeros in A . Then there exists an algorithm with input A that provides matrices B and N such that $A = B^{-1}N$ and $[B \ N]$ is an RIMD, or proves that A is not a network matrix. The time complexity of the algorithm is $O(n\alpha)$.*

Network matrices form the basis for all totally unimodular matrices. This was shown by a beautiful theorem of Seymour [46] (cf [47]). Not all totally unimodular matrices are network matrices or their transposes, as is shown by the matrices (cf Bixby [8]):

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad (2.3)$$

Seymour showed that each totally unimodular matrix arises, in a certain way, from network matrices and the matrices 2.3.

To describe the characterization theorem, first observe that total unimodularity is preserved under the following operations:

- (i) permuting rows or columns;

- (ii) taking the transpose;
- (iii) signing rows or columns;
- (iv) pivoting;
- (v) adding an all-zero row or column, or adding a row or column with one nonzero, being ± 1 ;
- (vi) repeating a row or a column;

Moreover, total unimodularity is preserved under the following compositions:

$$(vii) \quad A \oplus_1 B := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad (1\text{-sum});$$

$$(viii) \quad \begin{bmatrix} A & a \end{bmatrix} \oplus_2 \begin{bmatrix} b \\ B \end{bmatrix} := \begin{bmatrix} A & ab \\ 0 & B \end{bmatrix} \quad (2\text{-sum});$$

$$(ix) \quad \begin{bmatrix} A & a & a \\ c & 0 & 1 \end{bmatrix} \oplus_3 \begin{bmatrix} 1 & 0 & b \\ d & d & B \end{bmatrix} := \begin{bmatrix} A & ab \\ dc & B \end{bmatrix} \quad (3\text{-sum});$$

(here A and B are matrices, a and d are column vectors, and b and c are row vectors, of appropriate sizes). It is not difficult to see that total unimodularity is maintained under these compositions, e.g. with Ghouila-Houri's characterization (see [44]).

Theorem 2.6 [*Seymour's decomposition theorem for tot. uni. matrices.*] *A matrix A is totally unimodular if and only if A arises from network matrices and the matrices 2.3 by applying the operations (i), (ii), ... (ix). Here the operations (vii), (viii) and (ix) are applied only if both for A and for B we have: the number of rows plus the number of columns is at least 4.*

Finally, let A be a nonnegative connected network matrix, and suppose that we are given two row indexes, say ϵ and ρ ($\epsilon \neq \rho$). Let $G(A)$ be a basic network representation of A and q a path in $G(A)$ containing e_ϵ and e_ρ . If q passes through one of the edges e_ϵ and e_ρ forwardly and through the other one backwardly, then we say that e_ϵ and e_ρ are *alternating* in $G(A)$, otherwise *nonalternating*. If A has a basic network representation in which e_ϵ and e_ρ are alternating and an other one in which they are nonalternating, then N is said to be $\{\epsilon, \rho\}$ -*noncorelated*, otherwise $\{\epsilon, \rho\}$ -*corelated*.

For R a row index subset, an R -*network* representation denotes a network representation such that R is the index set of one basic edge. A matrix is called an R -*network* matrix if it has an R -network representation. A *basic* subgraph of a network representation $G(A)$, is a subgraph consisting of only basic edges.

Chapter 3

Bidirected graphs

In this section, we describe bidirected graphs. The notion of bidirected graphs was introduced by Edmonds [19] as a common generalization of both directed and undirected graphs. In a directed graph, if an endnode of an edge is its tail, then the other endnode of the edge must be its head. Undirected graphs can be viewed as graphs in which each edge has two heads. In a bidirected graph, endnode of an edge can be its head or tail, independently from each other. These graphs serve as a background for introducing binet matrices in Chapter 4.

Largely, we adopt the terminology and definitions given by Appa and Kotnyek [36], [6] and Zaslavsky [65]. Bidirected graphs have appeared in the literature several times. Schrijver [45] gave a necessary and sufficient condition for the existence of an integer solution to a linear inequality system $Ax \leq b$ in which A is the edge-node incidence matrix of a bidirected graph. Gerards and Schrijver [28] characterized bidirected graphs which lead to matrices with strong Chvátal rank 1.

3.1 Basic notions

A bidirected graph $G = (V, E)$ on node set $V = \{v_1, \dots, v_n\}$ and with edge set $E = \{e_1, \dots, e_m\}$ may have four kinds of edges. A *link* is an edge with two distinct endnodes; a *loop* has two identical endnodes; a *half-edge* has one endnode, and a *loose edge* has no endnode at all.

Every edge is signed with $+$ or $-$ at its endnodes. That is, links or loops can be signed with $+-$, $++$ or $--$; half-edges have only one sign, $+$ or $-$. If an edge is signed with $+$ at an endnode, then this node is an *in-node* or *head* of the edge. An endnode signed with $-$ is called the *out-node* or *tail* of the edge. One can think of the value $+$ as indicating that the edge is directed into the node, $-$ indicating direction away from the node. Figure 3.1 shows two possible graphical representations of the same bidirected graph. Heads and tails can be represented with signs as in (i), or arrows as in (ii). A link or loop with two different signs at its endnodes is said to be a *directed edge*. For simplicity, a directed edge is represented by only one arrow instead of two (see e_4 for example in Figure 3.1). All edges, except directed and loose edges, are called *bidirected edges*. If e_j is a link or a loop with endnodes v_i and $v_{i'}$ ($v_i = v_{i'}$ in the case of a loop), then we may denote e_j as $[v_i, v_{i'}]$, $]v_i, v_{i'}[$, $[v_i, v_{i'}[$ or $]v_i, v_{i'}]$ depending on the sign at the endnode; actually a closed (resp., open) bracket at v_i means that one sign at this endnode is $+$ (respectively, $-$). Similarly, a half-edge incident with v_i

may be denoted by $[v_i]$ or $]v_i[$. For instance in Figure 3.1, $e_1 =]v_1[$, $e_2 = [v_1, v_2]$, $e_3 = [v_2, v_2]$ and $e_4 =]v_2, v_3]$.

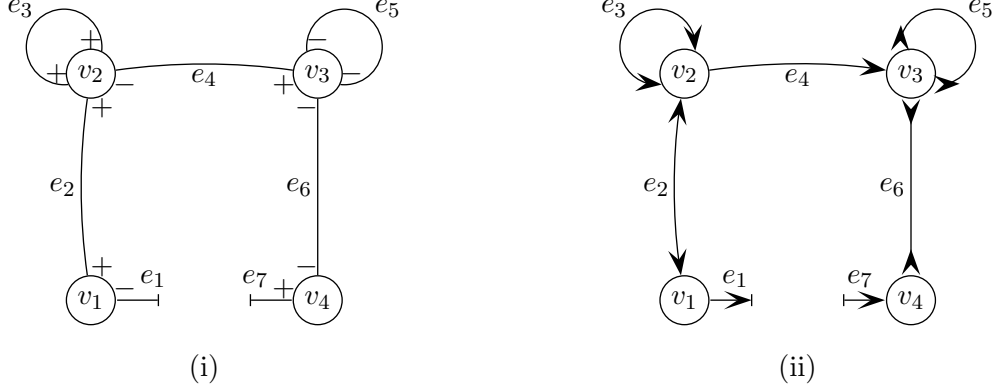


Figure 3.1: Possible graphical representations of a bidirected graph

Every edge e is given a *sign*, denoted by $\sigma_e \in \{+, -\}$. The signing convention we adopt is that the sign of a link or loop is $-$ times the products of the signs of its endnodes, the sign of a half-edge is always negative, and the sign of a loose edge is positive. Thus, a positive link or loop is a directed edge; as this is just like an ordinary directed graph, a directed graph is the same as a bidirected all-positive signed graph. The case of a positive loop arises when dealing with contraction of graphs, an operation defined later in the section. The normal case we encounter is that of a negative loop (where its endnode is either an in-node or an out-node). In Figure 3.1, the links e_2 and e_6 and loops e_3 and e_5 are negative, while the link e_4 is positive.

A *walk* in a bidirected graph is a sequence $(v_1, e_1, v_2, \dots, v_{t-1}, e_{t-1}, v_t)$ where v_i and v_{i+1} are endnodes of edge e_i ($i = 1, \dots, t-1$), including the case where $v_i = v_{i+1}$ and e_i is a half-edge. If the walk consists of only links and it does not cross itself, i.e. $v_i \neq v_j$ for $1 < i < t$, $1 \leq j \leq t$, $i \neq j$, then it is a *path*. In the bidirected graph of Figure 3.1, we have the path $(v_1, [v_1, v_2], v_2,]v_2, v_3], v_3,]v_3, v_4[, v_4)$. A closed walk which does not cross itself (except at $v_1 = v_t$) and goes through each edge at most once is called a *cycle*. So a loop, a half-edge or a closed path can make up a cycle. In Figure 3.1, there are exactly four cycles. The *sign of a cycle* is the product of the signs of its edges, so we have a *positive cycle* if the number of negative edges (or bidirected edges) in the cycle is even; otherwise, the cycle is a *negative cycle*. Obviously, a negative loop or a half-edge always makes a negative cycle. A *full cycle* in a bidirected graph is a cycle different from a half-edge.

A bidirected graph is *connected*, if there is a path between any two nodes. A *tree* is a connected bidirected graph which does not contain a cycle. A connected bidirected graph containing exactly one cycle is called a *1-tree*, indicative of the fact that a 1-tree consists of a tree and one additional edge. If the unique cycle in a 1-tree is negative, then we will call it a *negative 1-tree*. A *path-wheel* in a bidirected graph is a subgraph consisting of a negative cycle and a path from a node of the cycle to another one not in the cycle.

In a walk $(v_0, e_0, v_1, e_1, \dots, v_{t-1}, e_{t-1}, v_t)$ of G , a node v_i is *consistent* if the signs at the endnode v_i of edges e_{i-1} and e_i are different. (This definition applies to v_0 , with subscripts

modulo t , if $v_0 = v_t$ and $t > 0$.) A path is said to be *consistently oriented* if all nodes, except the first and last ones, are consistent. A *directed* path is a consistently oriented path with only directed edges.

3.2 Incidence matrix

In this section, we define the (*node-edge*) *incidence matrix* $In(G)$ of a bidirected graph G and called an IMB. We describe natural operations on $In(G)$ and the corresponding operations on G , and study linear independence and dependence of columns in $In(G)$. The rows and columns of $In(G)$ are identified with the nodes and edges of G , respectively. An entry (i, j) of $In(G)$ is 1 (resp., -1) if e_j is a link or a half-edge entering (resp., leaving) v_i , 2 (resp., -2) if e_j is a negative loop entering (resp., leaving) v_i , 0 otherwise. As an example, the node-edge incidence matrix of the bidirected graph depicted in Figure 3.1 is:

$$In = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

An *RIMB*, or restricted IMB, is an IMB with (linearly) redundant rows removed. The relationship of incidence matrices and bidirected graphs is two-way (or bidirectional). Given an $n \times m$ integral matrix In satisfying

$$\sum_{i=1}^n |(In)_{ij}| \leq 2 \text{ for } j = 1, \dots, m, \quad (3.1)$$

one can find a bidirected graph $G(M)$ with n nodes and m edges such that its node-edge incidence matrix is In . In other words, property (3.1) characterizes the node-edge incidence matrices of bidirected graphs.

Operations on $In(G)$ that maintain (3.1) can be translated to operations on bidirected graphs. Such operations are, for example, multiplying a row or column with -1 , or deleting a row or column. It can be easily verified that the following transformation also maintains (3.1).

$$In = \begin{bmatrix} \alpha & \mathbf{c} \\ \mathbf{b} & D \end{bmatrix} \rightarrow In' = D - \alpha \mathbf{b} \mathbf{c} \quad (3.2)$$

where α is a non-zero entry, \mathbf{b} is a column vector, \mathbf{c} is a row vector, and D is a submatrix of In .

We now give the graphical equivalents of these operations. They are extensions of standard operations on directed or undirected graphs, such as edge or node deletion, and edge contraction, taking into account the signs of the edges. When multiplying the j th column of In with -1 , we simply change the signs at the endnodes of edge e_j . In-nodes of the edge become out-nodes and vice versa, but the sign of the edge is unchanged. We call this operation *reversing the orientation* of an edge. Multiplying a row with -1 changes the signs at the corresponding endnode of all the incident edges. If the node was an in-node of an edge, it becomes its out-node and vice versa. Consequently, the sign of the incident links or

loops change, positive links or loops become negative and vice versa. We call this operation *switching* at a node.

Column deletion easily translates to bidirected graphs, it is equivalent to *deleting an edge* from the graph. Deletion of a row corresponds to the removal of the associated node together with edge-ends incident to the node. That is, links connected to this node become half-edges while loops and half-edges located at the deleted node become loose edges. All other edges and nodes remain unchanged. we call this operation *deleting a node* or *removing a node*.

Finally, the transformation (3.2) translates to a *contraction* in the bidirected graph. If e_1 is a negative loop or a half-edge, then $A' = D$, that is, edge e_1 and node v_1 are deleted from G . If e_1 is a positive link, then we get the ordinary graph contraction, that is, deletion of the edge and unification of its endnodes. If e is a negative link, we first switch at v_1 and then contract the now positive e_1 .

Let Q be a submatrix of the incidence matrix $In(G)$. It can be obtained by row and column deletions. By analogous operations, a bidirected graph $G(Q)$ can be obtained from G . It is clear that $G(Q)$ does not depend on the order of row and column deletions. This graph can be achieved by edge and node deletions, but strictly speaking, it is not a subgraph of G , as it can contain half-edges and loose edges that are not present in the original edge set E . However, when it does not create confusion, we will call it a subgraph of G .

In the rest of this section, we give a graphical characterization of linear independence and dependence in the node-edge incidence matrix of a bidirected graph. It is easily shown that columns of the matrix corresponding to a tree in the graph are linearly independent. However, contrary to the directed case, in bidirected graphs there are other linearly independent structures, as stated in Lemma 3.4. A subset C of edges in G is called a *circuit* if the columns of the matrix $In(C)$ form a minimal dependent set. The propositions listed below are needed to establish Lemma 3.4. We omit their proofs as they can be found in [36], or derived from very similar results appearing in, for example, [28, 61].

Proposition 3.1 *Let Q be a square matrix of size $n \geq 2$ whose nonzero elements are $q_{ii} = 1$ for $i = 1, \dots, n$; $q_{i+1,i} = -1$ for $i = 1, \dots, n-1$ and $q_{1n} = \pm 1$. That is, Q is of the following form*

$$Q = \begin{bmatrix} 1 & & & & \pm 1 \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & 1 & \\ & & & -1 & 1 \end{bmatrix} \quad (3.3)$$

Then $\det(Q) = 0$ if $q_{1n} = -1$ and $\det(Q) = 2$ if $q_{1n} = 1$.

Proposition 3.2 *Switchings at nodes do not change the sign of a cycle.*

It is easy to verify (and available as lemmas 3.1 in [61]) that any tree graph can be transformed to a directed graph by switching. More generally, this leads to:

Proposition 3.3 *A negative 1-tree can be transformed to a directed tree together with one extra bidirected edge forming a negative cycle with the tree by switchings.*

Lemma 3.4 *A square submatrix Q of the incidence matrix of a bidirected graph G is non-singular if and only if each connected component of $G(Q)$ is a negative 1-tree.*

This result is derived in [61], Theorem 5.1, in matroidal terminology. We give here the proof as in [6] without recourse to matroids.

Proof. It is enough to prove the theorem for the case where $G(Q)$ is a connected graph and Q has no zero column. Q is a square submatrix, so it corresponds to a 1-tree subgraph. Expanding $\det(Q)$ by rows corresponding to the pendant nodes of $G(Q)$, it is easy to establish that the singularity of Q depends on the singularity of its submatrix $In(C)$ corresponding to the cycle C of the 1-tree. If C is a loop or a half-edge, then it is obviously negative and $In(C)$ is non-singular. Otherwise, by permutations and row and column multiplications with -1 on $In(C)$, or equivalently by switchings and orientation reversals in C , $In(C)$ can be brought to a form of (3.3). It follows then from the previous propositions that C is negative if and only if $In(C)$ is non-singular. ■

This lemma has the following easy consequences.

Corollary 3.5 *Let In be a full row rank node-edge incidence matrix of a bidirected graph, and Q a collection of linearly independent columns of In . Then each connected component of $G(Q)$ either forms a tree or a negative 1-tree. Conversely, if every component of a subgraph $G(Q)$ forms a tree or a negative 1-tree, then the columns of Q are linearly independent.*

Corollary 3.6 *A circuit in a bidirected graph falls in one of the following categories.*

- (i) *it is a loose edge, or*
- (ii) *a positive cycle, or*
- (iii) *a pair of negative cycles with exactly one common node, or*
- (iv) *a pair of disjoint negative cycles along with a minimal connecting path.*

The latter two types are called *handcuffs*. For an illustration of Corollary 3.6, see Figure 3.2.

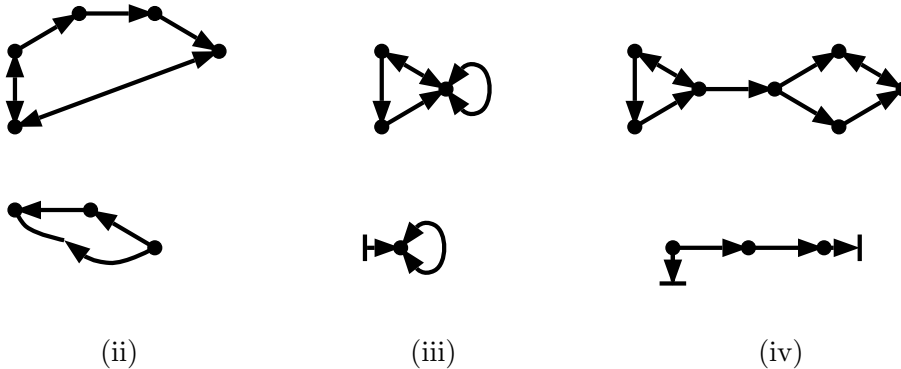


Figure 3.2: Examples of circuits.

Finally, Using Proposition 3.1 and Lemma 3.4, it follows that any nonsingular square submatrix of an IMB has a determinant equal to a power of 2 times ± 1 .

Theorem 3.7 *Let In be a full row rank node-edge incidence matrix of a bidirected graph. Then, for any nonsingular square submatrix Q of In , $\det(Q) = \pm 2^r$ for some $r \in \mathbb{N}$.*

Chapter 4

Binet matrices

This chapter is devoted to present known fundamental results on binet matrices which will be used frequently in later chapters. It describes the generalization of network matrices for bidirected graphs. Largely, the results and explanations are taken from [36, 4] and [11].

Network matrices can be defined in two equivalent ways. The graphical definition starts with a connected directed graph with a given spanning tree in it. The rows and columns of the network matrix are associated with the tree and non-tree edges, respectively. For any non-tree edge f , we find the unique cycle (called the fundamental cycle) which contains f and some edges from the tree. The column of the network matrix corresponding to f will contain ± 1 in the rows of the tree edges in its fundamental cycle and 0 elsewhere. The signs of the non-zeros depend on the directions of the edges. If walking through the tree along the fundamental cycle starting at the tail of f , a tree edge lies in the same direction, it gets a positive sign, if it lies in the opposite direction, it gets a negative sign.

In the algebraic derivation of the network matrices, the incidence matrix In of the directed graph is used. To make it full row rank, an arbitrary row is deleted. Every basis in this full row rank matrix In' corresponds to a spanning tree in the graph. If basis B is associated with the given spanning tree, and we denote the remaining part of In' as N , then the network matrix equals $B^{-1}N$.

We apply these methods to bidirected graphs to get the bidirected analogue of network matrices, the *binet matrices*¹. We define binet matrices in the algebraic way but, in parallel with network matrices, we also provide an algorithm to determine the columns of a binet matrix using its graphical representation. Similarly to network matrices, if the graphical definition is not available, then the analysis of binet matrices would be more cumbersome. That is why we will spend a relatively large part of the chapter on explaining the graphical method of deriving binet matrices. Most of Section 4.1 is devoted to this task.

In Section 4.2, we describe operations on binet matrices. Section 4.3 introduces some particular binet representations. In Section 4.4 we discuss linear and integer programming with an RIMB or a binet matrix as constraint matrix. Section 4.5 is about the class of 2-regular matrices which contains all binet matrices. Section 4.6 deals with matroids. To exhibit the connection of binet matrices to existing special classes of matroids, we introduce signed

¹the term binet is used as a short form for *bidirected network*, but by coincidence it also matches the name of Jacques Binet (1786-1856) who worked on the foundations of matrix theory and gave the rule of matrix multiplication.

graphs and matroids in Subsections 4.6.1 and 4.6.2, respectively. Then the signed-graphic matroid based on signed graphs is defined in Subsection 4.6.3.

4.1 Definition and graphical representation

We first define binet matrices algebraically and then show that there exists an equivalent graphical definition.

Definition. A matrix A is called a *binet matrix* if there exist a full row rank incidence matrix In of size $n \times m'$ of a bidirected graph G and a basis B of it such that $In = [B \ N]$ and $A = B^{-1}N$.

From this definition, since any RIMD is an RIMB, it follows that any network matrix is a binet matrix.

Edges in the subgraph $G(B)$ of G are called *basic* edges. The edges of G that are not in $G(B)$ (i.e., those of $G(N)$) are the *nonbasic* edges. By Lemma 3.4, the graph $G(B)$ consists of negative 1-tree components. The unique cycle in a basic component is called a *basic cycle*. Basic edges, respectively nonbasic ones, are in one-to-one correspondance with rows of A , respectively columns of A .

The bidirected graph G with the indication of basic and nonbasic edges is called a *binet representation* of A (not unique in general) and is denoted $G(A)$. Let $m = m' - n$. Basic edges of the binet representation will be noted e_1, e_2, \dots, e_n and the nonbasic ones f_1, f_2, \dots, f_m , so that e_i ($1 \leq i \leq n$), respectively f_j ($1 \leq j \leq m$), corresponds to the i th row, respectively j th column of A . Sometimes, we will identify some row or column of A with its corresponding basic or (respectively) nonbasic edge. See the binet matrix (4.1) and Figure 4.1.

The same binet matrix may arise from different incidence matrices, that is, it may have different binet representations. For example, the two bidirected graphs in Figure 4.1 give two possible binet representations of binet matrix A below, as can be checked by writing out the incidence matrix of the graphs, taking the inverse of the basis B (corresponding to the edges e_1, e_2 and e_3) and multiplying it with N (corresponding to the edges f_1, f_2 and f_3).

$$A = \begin{array}{c|ccc} & f_1 & f_2 & f_3 \\ \hline e_1 & 1 & 0 & 1 \\ e_2 & 1 & 1 & 0 \\ e_3 & 0 & 1 & 1 \end{array} \quad (4.1)$$



Figure 4.1: Different binet representations of binet matrix (4.1).

Let $1 \leq j \leq m$. Since $A_{\bullet j} = B^{-1}N_{\bullet j} \Leftrightarrow BA_{\bullet j} = N_{\bullet j}$, the column $A_{\bullet j}$ represents the unique coordinates of vector $N_{\bullet j}$ in the basis given by B . Let $\{B_{\bullet k_1}, B_{\bullet k_2}, \dots, B_{\bullet k_t}\}$ be the subset of columns of B where the coordinates are non-zero and $B' = [N_{\bullet j}, B_{\bullet k_1}, \dots, B_{\bullet k_t}]$. The columns of the matrix B' form a minimal dependent set in \mathbf{R}^n . The subgraph $G(B')$ of G is called the *fundamental* circuit of f_j . A *basic* subgraph of a binet representation $G(A)$ is a subgraph consisting of only basic edges. The *basic* fundamental circuit of f_j is the fundamental circuit of f_j without f_j . If the fundamental circuit of f_j is a handcuff of type (iii) (see Corollary 3.6), then it is called a *pathcycle*.

By Corollary 3.6, we can describe the different types of fundamental circuits according to f_j . Later (see Lemma 4.9), we will prove that a binet matrix has always a binet representation in which each component of $G(B)$ has exactly one bidirected edge (contained in the basic cycle). So, in what follows, we assume that each component of $G(B)$ has exactly one bidirected edge contained in the basic cycle. Let v, v' be the end-nodes of f_j ($v = v'$ if f_j is a half-edge or loop). The vertex v is contained in some maximal negative 1-tree, call it T . Let p be the path in T from v to the first node, say w , in the basic cycle ($v = w$ if v lies on the basic cycle). In the same way, define T', p', w' and q' with respect to v' . Each of the paths p and p' is called a *stem* issued from f_j . A subpath of a stem is called a *substem*. If $T \neq T'$, then f_j is called a *2-edge*, otherwise a *1-edge*.

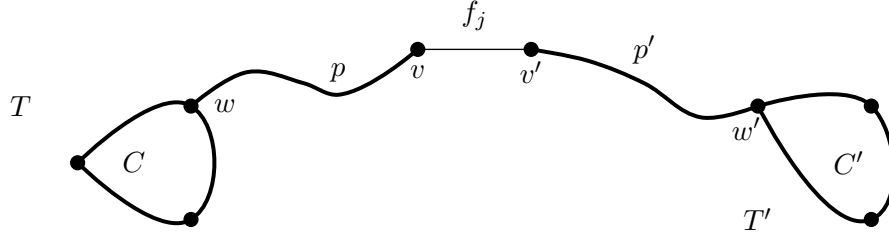


Figure 4.2: An illustration of the fundamental circuit of f_j in case $T \neq T'$.

Assume $T \neq T'$. Let C and C' be the (negative) cycles in T and T' , respectively. The fundamental circuit of f_j is the bidirected graph formed by f_j, p, C, p' and C' . See Figure 4.2.

Now assume $T = T'$. Let C_1 (respectively, C_2) be the path in the basic cycle from w to w' containing the bidirected edge (respectively, only directed edges). If f_j is bidirected, then the fundamental circuit of f_j is the subgraph of G consisting of f_j, p, C_1 and p' . If f_j is directed and $w \neq w'$ then the fundamental circuit of f_j is the subgraph of G consisting of f_j, p, C_2 and p' . If f_j is directed and $w = w'$, then the fundamental circuit of f_j is the bidirected graph whose edge set contains f_j , the edges of p not in p' and the edges of p' not in p . See Figure 4.3 for an illustration.

A *minimal covering closed walk* of a circuit C in G is a closed walk of minimal length that covers each edge and is denoted by $w(C)$. Using Corollary 3.6, we deduce that the walk $w(C)$ covers each edge of a connecting path twice (provided that C is a handcuff) and each other edge exactly once. For any edge f in C , the walk $w(C)$ is said to be *consistently oriented according to f* if every node in $w(C)$, except the endnodes of f , is consistent. It is easy to verify that for any edge f in C , one may orient the edges of $C \setminus \{f\}$ so that $w(C)$ becomes consistently oriented according to f . (One may consult [63] for a detailed discussion of how

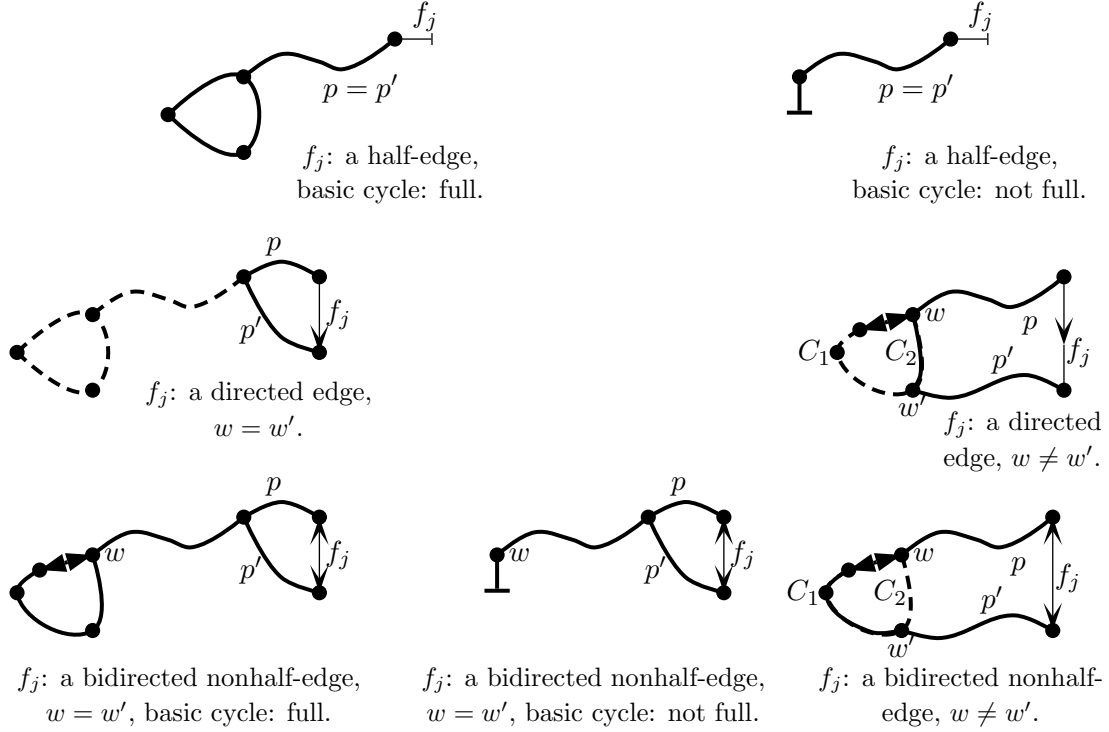


Figure 4.3: An illustration of the different types of fundamental circuit of a nonbasic edge f_j in case $T = T'$. (The fundamental circuit of f_j is formed by f_j and heavy edges.)

to orient a signed graph. We note that in articles by Zaslavsky the word used for "consistent" is "coherent", while in [6], the corresponding term is "incoherent".)

In what follows, we give a graphical method adapted from [11] to obtain a binet matrix from its binet representation. This spares us the need to compute the inverse of the basis, makes handling of binet matrices much easier, and helps in proving properties of binet matrices in later sections.

Procedure WeightCircuit

Input: A bidirected graph G , a basis B of the incidence matrix $In(G)$, and a non-zero column $N_{\bullet j}$ of In not in B .

Output: The column $A_{\bullet j} = B^{-1}N_{\bullet j}$.

- 1) let T be the edge set corresponding to the basis B and C the fundamental circuit of f_j in G and $w(C)$ a minimal covering closed walk of C ;
 - 2) reorient edges of $C \setminus \{f_j\}$ so that $w(C)$ becomes consistently oriented according to f_j ;
 - 3) assign weights +1 to each singly covered edge (except half-edges), +2 to each doubly covered edge or half-edge, and 0 to the other edges in T ;
 - 4) negate the values assigned to edges that were reoriented; divide by 2 if necessary to ensure that f_j has weight +1;
- output the edge weights on T that correspond to the different coefficients of $A_{\bullet j}$.

From this procedure, it is easy to deduce the following two lemmas.

Lemma 4.1 *For any nonbasic edge f_j and e_i a basic edge in the fundamental circuit of f_j , we have*

$$A_{ij} = \begin{cases} \pm \frac{1}{2} & \text{if } f_j \text{ is a 2-edge or a half-edge, and } e_i \text{ is in a full basic cycle;} \\ \pm 2 & \text{if } f_j \text{ is a bidirected nonhalf 1-edge, and } e_i \text{ is a half-edge or belongs to} \\ & \text{both stems issued from } f_j; \\ \pm 1 & \text{otherwise.} \end{cases}$$

Lemma 4.2 *Let f_j be a nonbasic edge, C its fundamental circuit and $w(C)$ a minimal covering closed walk of C . Then $w(C)$ is consistently oriented according to f_j if and only if $A_{\bullet j}$ is nonnegative.*

From these lemmas, we can describe the different types of fundamental circuits of a nonbasic edge f_j with weights on the edges. By assuming that each component of $G(B)$ has exactly one bidirected edge (which is entering), an illustration is given in Figure 4.4 (where p, p', C, C', C_1 and C_2 are defined at page 41).

4.2 Operations on binet matrices

In this section we give some operations that, when applied to a binet matrix, result in another binet matrix. The content of the section can be found in [6, 36].

Let us start with a graphical operation which does not change the binet matrix. Switching at a node of a binet representation keeps the parity of the number of negative edges in a fundamental circuit (see Proposition 3.2), so clearly does not affect the calculations (see the procedure `WeightCircuit` in Section 4.1). The matrix operation equivalent to switching is multiplying a row of the node-edge incidence matrix In by -1 . The effect of this change is easy to detect: a column in the inverse of the basis B and the corresponding row in the nonbasic part N is multiplied by -1 . So $B^{-1}N$ does not change.

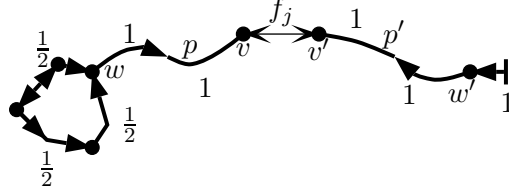
Lemma 4.3 *Switching at a node of a binet representation of a binet matrix keeps the binet matrix unchanged.*

Permuting rows or columns of a binet matrix obviously results in another binet matrix. Define a *signing* of a vector as multiplying it by -1 . Signing a row or a column of a binet matrix also results in a binet matrix, as it is equivalent to reversing the orientation of the corresponding basic, or nonbasic, edge.

Lemma 4.4 *Let A be a binet matrix. Matrix A' obtained by the following operations from A is also a binet matrix.*

- a) deleting a row or a column,
- b) repeating a row or a column,
- c) signing a row or a column,
- d) adding a unit row or a unit column.

The case $T \neq T'$:



The case $T = T'$:

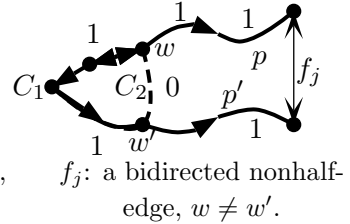
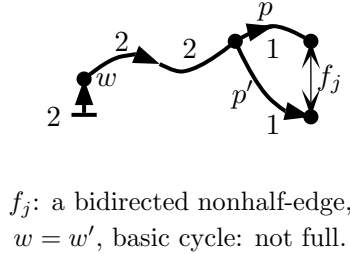
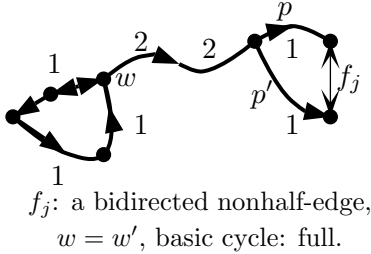
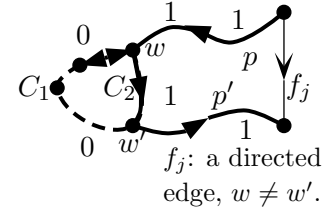
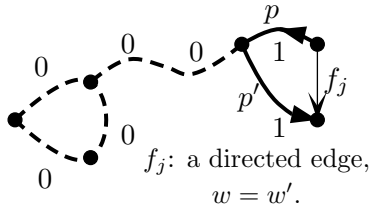
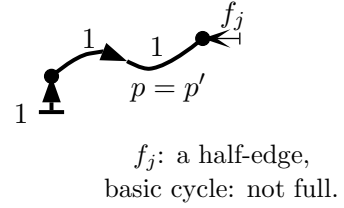
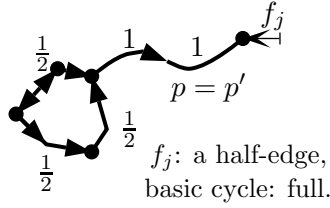


Figure 4.4: An illustration of the different types of fundamental circuit of a nonbasic edge f_j with weights on the edges, when A is nonnegative. Each basic 1-tree contains exactly one bidirected edge (which is entering). (In case $T \neq T'$, the basic full cycle may be replaced by a half-edge and the half-edge by a full cycle.)

The proof of Lemma 4.4 can be found in [6]. Part a) of Lemma 4.4 can be rephrased.

Theorem 4.5 *Every submatrix of a binet matrix is binet.*

Using Theorem 4.5, we may deduce Lemma 4.6 below. This lemma will enable us to assume that a given matrix A is connected for the recognition problem discussed in Chapter 6.

Lemma 4.6 *A matrix which may be decomposed into blocks is binet if and only if all blocks are binet.*

Another operation that preserves binetness is pivoting.

Lemma 4.7 *Let A be a binet matrix. Matrix A' obtained by pivoting on a non-zero element of A is binet.*

Proof. The proof follows from the fact that pivoting is equivalent to changing the basis. ■

Lemma 4.8 *Let A be a connected matrix. The matrix A' obtained by pivoting on a non-zero element of A is connected.*

Proof. The proof by contradiction is straightforward. ■

Switching at a node of the binet representation of a binet matrix keeps the binet matrix unchanged, which provides a method to find a special binet representation of a binet matrix.

Lemma 4.9 *There always exists a representation of a binet matrix where each connected component of the basis contains only one bidirected edge.*

Finally, we show that the number of nonequal columns of a binet matrix is bounded by a quadratic function of the number of rows.

Lemma 4.10 *Let A be a binet matrix of size $n \times m$ having no two identical columns. Then*

$$m \leq 4 \binom{n}{2} + 2n + 1.$$

Proof. Let $G(A)$ be a binet representation of A . We know that $G(A)$ contains n nodes. So, since no two columns in A are identical, the number of distinct nonbasic directed links is at most $2 \binom{n}{2}$ (which is two times the number of pairs of nodes). Similarly, for nonbasic bidirected links. Moreover, the number of nonbasic half-edges is bounded by $2n$ and there is at most one loose edge (whose corresponding column is zero). This concludes the proof. ■

4.3 Some binet representations

In this section, we define some particular binet representations and related results. Given a binet representation of a matrix A , by removing the nonbasic edges we obtain a union of basic components which is called a *basic binet representation* of A . Using A and a basic binet representation of A , it is easy to construct a binet representation of A .

A binet representation of A is called *proper* if each basic component has exactly one bidirected edge (contained in the basic cycle), this one is entering, and one end-node of the basic bidirected edge is not a consistent node of the basic cycle. A node in the basic cycle which is not consistent is called a *central node*. A *central edge* is an edge in a basic cycle incident with a central node.

Lemma 4.11 *If the matrix A is binet, then it has a proper binet representation.*

Proof. Consider a binet representation $G(A)$ of A . We may suppose that $G(A)$ has exactly one basic maximal negative 1-tree, call it T , containing a full basic cycle C which is not a loop. By Lemma 4.9, we may suppose that exactly one edge of T is bidirected and this edge is entering. Denote by w_1, \dots, w_ρ the vertices of the cycle in clockwise order and assume that $[w_1, w_\rho] \in G(A)$. Consider the longest directed path from w_1 to a vertex, say w_i , in C . If $i = 1$ or ρ , then one directed edge of the cycle is entering an end-node of the bidirected edge. Otherwise, we have directed edges $]w_{j-1}, w_j]$ for $j = 2, \dots, i$ and $]w_{i+1}, w_i]$ in $G(A)$. By switching at all nodes of the trees in $G(A) \setminus E(C)$ containing some node w_j with $1 \leq j \leq i-1$, we obtain a new binet representation $G'(A)$ of A such that $[w_{i-1}, w_i],]w_{i+1}, w_i] \in G'(A)$, and $[w_{i-1}, w_i]$ is the unique bidirected edge in the maximal basic 1-tree. ■

A $\frac{1}{2}$ -binet representation is a proper binet representation in which every basic cycle is a half-edge. A matrix is said to be $\frac{1}{2}$ -binet if it has a $\frac{1}{2}$ -binet representation.

A *cyclic representation* of a matrix is a proper binet representation of the matrix having exactly one basic cycle, and this one is full. A matrix is said to be *cyclic*, if it has a cyclic representation. For R a row index set, an *R -cyclic representation* of a matrix A denotes a cyclic representation such that R is the edge index set of the basic cycle, and this cycle is contained in the fundamental circuit of at least one nonbasic edge f_j (or equivalently $R \subset s(A_{\bullet j})$ for at least one column index j of A). A matrix is said to be *R -cyclic*, if it has an R -cyclic representation. If a matrix A has an R -cyclic representation $G(A)$, up to row permutations we may assume that w_1, \dots, w_ρ are the vertices of the basic cycle, $e_1 = [w_1, w_\rho]$, $e_\rho =]w_{\rho-1}, w_\rho] \in G(A)$ and for $i = 2, \dots, \rho$, e_i is the edge incident with w_{i-1} and w_i .

A *bicyclic representation* of a matrix is a proper binet representation of the matrix having exactly two basic cycles, and these are full. A matrix is said to be *bicyclic*, if it has a bicyclic representation.

Finally, we say that a binet representation is an $\{\epsilon, \rho\}$ -central representation, if it is cyclic, e_ϵ and e_ρ are edges of the basic cycle incident with one common central node, and one of the edges e_ϵ and e_ρ is bidirected. Let us mention here that if a binet representation is $\{\epsilon, \rho\}$ -central and e_ρ is bidirected, then up to switching operations it is possible to transform it into an $\{\epsilon, \rho\}$ -central representation such that e_ϵ is bidirected. We say that a matrix is an $\{\epsilon, \rho\}$ -central matrix, if it has an $\{\epsilon, \rho\}$ -central representation. Suppose that a matrix A has an $\{\epsilon, \rho\}$ -central representation $G(A)$ and T is the basic maximal 1-tree in $G(A)$. We denote by $G_1(A)$ the connected component of $T \setminus \{e_\epsilon, e_\rho\}$ containing the central node incident to e_ϵ and e_ρ , and by $G_0(A)$ the other connected component. We say that a basic subgraph of $G(A)$ is *on the right* (respectively, *on the left*) of $\{e_\epsilon, e_\rho\}$, if it is contained in $G_1(A)$ (respectively, $G_0(A)$).

Using Lemma 4.1, we may deduce the following important lemma.

Lemma 4.12 *Let A be an integral matrix. Then A is binet if and only if it is either cyclic or $\frac{1}{2}$ -binet.*

4.4 Linear and integer programming

In this section, we deal with linear and integer programming in which the constraint matrix is the node-edge incidence matrix of a bidirected graph (IMB). Let us designate a linear (respectively, integer) programming problem with an IMB constraint matrix a *bidirected LP*

(respectively, *bidirected IP*). It is shown that efficient methods are available to solve bidirected LPs and IPs. Finally, we see that the problem of converting some linear or integer program to a bidirected LP or IP, respectively, using elementary row operations on the constraint matrix is equivalent to the recognition of binet matrices.

Consider the linear program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & 0 \leq x \leq \alpha \end{aligned} \tag{4.2}$$

where $c, \alpha \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times m}$ for some $n, m \in \mathbb{N}$ and some entries of α might be infinite.

If A is an RIMB, then problem (4.2) is a bidirected LP, and (4.2) can be seen as a generalization of the transshipment problem described at page 30. The optimal solution of a bidirected LP can be achieved by general-purpose methods, like the simplex algorithm, or the strongly polynomial algorithm of Tardos [52]. This latter states that for rational linear programming problems in which the elements of the constraint matrix are bounded, there exists an algorithm to solve the problem that uses arithmetic operations whose number is a polynomial function of the dimension of the problem and which act on rationals of size polynomially bounded by the size of the input. More about this ingenious algorithm can be found in [44]. Since the elements of an RIMB are between -2 and 2 , Tardos' algorithm on bidirected LP has a strongly polynomial worst-case running time. However, despite this attractive theoretical complexity result, the up-to-date implementations of the simplex algorithm usually outperform Tardos' strongly polynomial method on practical instances (see [36]).

Alternatively, Kotnyek [36] proved that a bidirected LP can be converted to a generalized network problem. A *generalized network* is defined on a connected digraph G . There is a real non-zero multiplier p_e associated with each edge $e = (i, j)$ of G . We assume that if a unit flow leaves the tail i of e , then p_e units arrive at j . G can also contain loops, i.e., edges whose tail and head coincide. It is assumed that the multiplier of a loop cannot be $+1$, as it would mean that the same flow leaves and enters the node on such a loop, making the loop redundant. Trivially, if all multipliers are equal to 1, then we have the well-known pure network.

A way of describing a generalized network is with its node-edge incidence matrix. The column of the incidence matrix that corresponds to a non-loop edge $e = (i, j)$ has -1 in row i and p_e in row j , zeros elsewhere. If e is a loop at node i , then its column has only one non-zero, $(p_e - 1)$ in row i . A *generalized network problem* is a linear program in which the constraint matrix is the node-edge incidence matrix of a generalized network. See [3, 39].

The idea behind the network simplex method is that the main steps of the simplex method (such as calculating the primal and dual solutions corresponding to a basis, or changing the basis) can be executed on the network associated with the constraint matrix. This idea can be followed in generalized networks too, leading to the *generalized network simplex method*. It should be noted that the generalized network simplex method is not polynomial in the worst case, but for most of the practical problems it is much more efficient than the simplex method, or the strongly polynomial method mentioned above (see the reference notes in [3]).

Now we turn to the integer case, that is, we are facing with the following integer program:

$$\begin{aligned}
 \min \quad & c^T x \\
 \text{s.t.} \quad & Ax = b \\
 & 0 \leq x \leq \alpha, \\
 & x \text{ integral}
 \end{aligned} \tag{4.3}$$

where $c, \alpha \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times m}$ for some $n, m \in \mathbb{N}$ and some entries of α might be infinite.

If A is an RIMB, then (4.3) is a bidirected IP. The bidirected IP problem was introduced by Edmonds [20] as a generalization of the following problem. Given any graph $G = (V, E)$ with a real numerical weight c_e for each edge $e \in E$ and an integer b_v for each node $v \in V$, find in G , if there is one, a subgraph G' which has degrees b_v at nodes v and whose edges have maximum weight-sum. This is called the *optimum b -matching problem* on graph G . A *b -matching* E' in G is a subset $E' \subseteq E$ of edges such that b_v edge-ends of edges in E' meet node v . Obviously there is a 1 – 1 correspondence between the b -matchings in a graph G and the b -degree subgraphs of G that contain all the nodes of G .

If A is an IMB with respect to a graph G (all edges are bidirected entering links) and $\alpha_e = 1$ for every edge in G , then (4.3) is simply the optimum b -matching problem relative to G . Edmonds showed that any bidirected IP is equivalent to a b -matching problem (see also [37]). Moreover, Edmonds and Johnson [21] proved that there exists a polynomial algorithm to solve b -matching problems, even on bidirected graphs.

At last, suppose that A is in standard form (i.e., the first n columns of A constitute an identity matrix) and we wish to convert the problem (4.2) (resp., (4.3)) to a bidirected LP (resp., IP) using only elementary row operations on A . This is equivalent to the problem of recognizing binet matrices as stated in the next theorem.

Theorem 4.13 *Let A be a real matrix in standard form. Then A is projectively equivalent to an RIMB if and only if A is a binet matrix.*

Proof. The proof is similar to the proof of Theorem 2.4. ■

By Theorems 4.13 and 6.1, it results that there is an $O(\max(n^6 m, n^2 m^3))$ algorithm (the algorithm Binet) for determining whether A is projectively equivalent to an IMB, and thus whether the problem (4.2) (resp., (4.3)) is convertible to a bidirected LP (resp., IP). If such a conversion is possible, the algorithm Binet, in effect, carries it out.

4.5 A subclass of 2-regular matrices

In this section we introduce the definition of k -regular matrices given by Appa and Kotnyek [5, 36], as a generalization of totally unimodular matrices, and provide a geometrical interpretation. Then we concentrate on the case $k = 2$. It is proved that 2-regular matrices contain binet matrices. Moreover, we present some known theorems about binet matrices or their transposes with strong Chvátal rank 1. Finally, examples of minimally non-binet 2-regular matrices are given. The content of this section is mainly taken from [36].

Given a rational matrix A of size $n \times m$ and a rational vector b of size n , consider $P = \{x : x \geq 0, Ax \leq b\}$. A primary problem of integer programming is to find the *integer hull* $P_I = \text{conv}\{P \cap \mathbb{Z}^m\}$ of the polyhedron P . A rephrasing of Theorem 2.3 is that in the case of an integral matrix A , we have $P = P_I$ for all integral right hand side vectors b , if and only if A is totally unimodular. In terms of integer programming, totally unimodular matrices are the integral matrices for which $\max\{cx \mid Ax \leq b, x \geq 0\}$ has integral optimal solutions for any c and any integral b .

There are situations, however, that take us beyond total unimodularity. Theorem 2.3 holds provided that A is integral. What if the matrix in question is not integral? Or what can we say about matrices A that ensure integral optimal solutions for only a special set of right hand side? These questions are not independent. If A is rational, then one can find a nonnegative integer k , such that if we multiply every row of A by k , we get an integral matrix, kA . But then instead of inequalities $Ax \leq b$, we have $kAx \leq kb$ and we deal with polyhedra that are required to be integral for only special b' vectors, namely for those whose elements are integer multiples of k . For example, if $k = 2$, so the elements of A are halves of integers, then we are to characterize integral matrices A' for which $\{x \mid A'x \leq b', x \geq 0\}$ is integral for all **even** vectors b' . Or equivalently, we examine matrices that provide half-integral vertices for polyhedra with integral right hand sides.

A matrix is called *k-regular* ($k \in \mathbb{N}$) if for each of its non-singular square submatrices π , $k\pi^{-1}$ is integral. *k-regularity* is the property that takes over the role of total unimodularity in the theory of rational matrices that ensure integral vertices for polyhedra with special right hand sides. One important theorem states this (see [36]).

Theorem 4.14 *A rational matrix A is k-regular, if and only if the polyhedron $\{x \mid Ax \leq kb, x \geq 0\}$ is integral for any integral vector b .*

The second nice result is the following.

Theorem 4.15 *Every binet matrix is 2-regular.*

Proof. Let A be a binet matrix of size $n \times m$. By Theorem 4.5, we only have to prove that for any non-singular $n \times n$ submatrix π of A the matrix $2\pi^{-1}$ is integral. Let π be a non-singular $n \times n$ submatrix of A . Then, by definition of a binet matrix, there exists an RIMB $In = [B \ B' \ N]$ where B and B' are two disjoint bases of In , $A = B^{-1}[B' \ N]$ and $\pi = B^{-1}B'$. Thus $(B')^{-1}B = \pi^{-1}$ is also binet. By Lemma 4.1, every entry of π^{-1} is half-integral, which completes the proof. ■

Now let us define the notion of strong Chvátal rank 1. Given a polyhedron P , in several theoretical and practical problems, we have $P \neq P_I$. To tackle these cases and find integer solutions, different methods have been developed. One approach is the cutting plane method, pioneered by Gomory [30]. Its most basic concept is the *Chvátal-Gomory (CG) cut*, defined as follows. Given a rational $n \times m$ matrix A and a rational vector b of size n , a CG cut of the polyhedron $P = \{x : Ax \leq b\}$ is an inequality of the form $\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$ where $\lambda \in \mathbb{R}_+^n$ and $\lambda^T A \in \mathbb{Z}^m$ ($\lfloor \cdot \rfloor$ denotes the lower integer part).

The intersection of P with the half-spaces induced by all possible CG cuts is its *rank-1 closure*, denoted P_1 . Obviously, $P_I \subseteq P_1 \subseteq P$. It is also known that $P = P_I$ holds if and only

if $P = P_1$. Matrices A for which the integer hull P_I of $P = \{x : x \geq 0, Ax \leq b\}$ is the same as the rank-1 closure P_1 for any integral b are said to have *Chvátal rank 1*.

A stronger requirement is to assume that we have lower and upper bounds on Ax and x , so the polyhedron is of the form $P = \{x | \alpha \leq x \leq \beta, a \leq Ax \leq b\}$. Matrix A has *strong Chvátal rank 1*, if $P_1 = P_I$ for any integral choice for α, β, a and b (including ∞).

Edmonds and Johnson [22, 21] showed that if A is the node-edge incidence matrix of a bidirected graph, then it has strong Chvátal rank 1. (That is why matrices with strong Chvátal rank 1 are sometimes said to have the *Edmonds-Johnson property*.) In [28], Gerards and Schrijver gave a characterization of matrices that are edge-node incidence matrices of bidirected graphs and have strong Chvátal rank 1. The key matrix in their characterization is $In(K_4)$, the edge-node incidence matrix of K_4 , the complete graph on four nodes:

$$In(K_4) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The edge-node incidence matrices of bidirected graphs are exactly the integral matrices In (of size $m \times n$) that satisfy the transposed version of (3.1), namely,

$$\sum_{j=1}^n |(In)_{ij}| \leq 2 \text{ for } i = 1, \dots, m, \quad (4.4)$$

The characterization appearing in [28] is now the following.

Theorem 4.16 (*Gerards and Schrijver*) *An integral matrix satisfying 4.4 has strong Chvátal rank 1, if and only if it cannot be transformed to $In(K_4)$ by a series of following operations:*

(i) *deleting or permuting rows or columns, or multiplying them by -1 ;*

(ii) *replacing matrix $\begin{bmatrix} 1 & g \\ f & D \end{bmatrix}$ by the matrix $D - fg$.*

Here we extend the set of matrices with strong Chvátal rank 1 by stating that integral binet matrices are such [36].

Theorem 4.17 *If A is an integral binet matrix, then it has strong Chvátal rank 1.*

Note that Theorem 4.17 cannot be extended to rational binet matrices, as the following example shows.

$$A = B^{-1}N = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 1 & 1 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \text{ with } B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Binet matrix A does not have strong Chvátal rank 1, because the non-zero integral solutions of the polyhedron $P = \{x : 0 \leq x \leq 1, 0 \leq Ax \leq 1\}$ are : $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$,

so $x_1 + x_2 + x_3 \leq 1$ is a facet of P_I . But $(1, \frac{1}{2}, \frac{1}{2}) \in P$, so $\delta = 2$ is the smallest value for which $x_1 + x_2 + x_3 \leq \delta$ is valid for P . (In [36] the notion of half-integral cut is discussed.)

Now let us study some minimally non-binet matrices. The graphical equivalent of the matrix operations in Theorem 4.16 can also be given, following [28] and Section 4.2. Deleting a row or a column of an edge-node incidence matrix is equivalent to deleting an edge or a node from the graph. Multiplying a row with -1 translates to reversing the direction of an edge, while multiplying a column with -1 corresponds to a switching. Operation (ii) has already appeared as (3.2), so it is the same as the contraction of an edge (note that (3.2) is symmetric to transposing). Thus, the edge-node incidence matrix of a bidirected graph has strong Chvátal rank 1, if and only if the graph cannot be transformed to K_4 by a series of edge and node deletions, edge direction reversals, switching and contractions. Combining Theorems 4.16 and 4.17 we get:

Theorem 4.18 *If a bidirected graph can be transformed to K_4 by a series of edge and node deletions, edge direction reversals, switching and contractions, then its edge-node incidence matrix is not binet.*

Furthermore, let $In(K_6)$ the edge-node incidence matrix of the complete graph on six nodes. Kotnyek proved that we cannot change the signs of some entries of $In(K_6)$ to make it binet. On the other hand, if the edges of K_6 are oriented so that the graph is directed, then the corresponding edge-node incidence matrix is the transpose of a network matrix, so it is totally unimodular. Thus, we have an example of a matrix which is totally unimodular, but not binet. A complete list of minimally non-binet totally unimodular matrices follows from a submitted paper of Hongxun Qin, Daniel C. Slilaty and Xiyngqian Zhou [43].

There are also minimally non-binet 2-regular matrices that are not totally unimodular. A trivial example is $[\frac{1}{2} 2]$. No binet matrix can have a ± 2 and a $\pm \frac{1}{2}$ in the same row or column, for example because then pivoting on the $\pm \frac{1}{2}$ would result in a ± 4 . The matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

and its transpose are minimally non-binet, as can be shown by careful analysis of all the cases, but they are 2-regular and clearly not totally unimodular.

At last, we saw in Section 4.1 that binet matrices extend the class of network matrices, and in Section 2.3 that every totally unimodular matrix is built up by simple operations from network matrices and two further matrices (2.3) that are not network. We mention here that the matrices (2.3) were proved to be binet by Kotnyek [36]. Since the class of 2-regular matrices extends the binet matrices, it is conjectured that a characterization theorem should exist for a decomposition of 2-regular matrices into binet matrices and probably other non-binet matrices.

4.6 Matroids related to binet matrices

In Subsection 4.6.1, we introduce a generalization of undirected graphs, called signed graphs. Signed graphs can be achieved from bidirected graphs by ignoring the signs at the ends of the edges and focusing on only whether an edge is bidirected or directed. Subsection 4.6.2

contains an overview of the necessary theory about matroids. In Subsection 4.6.3, we define the signed-graphic matroid, that is a combinatorial structure associated with a signed graph. We will show that binet matrices are representative matrices of signed-graphic matroids, therefore results about bidirected graphs have consequences in terms of these matroids. We define also the near-regular matroid as a generalization of regular matroids.

The most important reference about signed graphs is Thomas Zaslavsky's work. His research is the basis of this chapter. Signed graphs were introduced by Harary [32]. Zaslavsky's annotated bibliography of signed and gain graphs [64], which contains hundreds of references, is an essential tool for anyone interested in the subject. Our notations and results mainly follow the Glossary of Signed and Gain Graphs [65], by Zaslavsky, and Kotnyek's thesis [36].

4.6.1 Signed graphs

A *signed graph* is a pair $\Sigma = (G, \sigma)$, where G is an undirected graph with possibly loose edges (with no endnodes) and half-edges (having exactly one endnode), and the edges of G are labelled by $+$ or $-$, that is, there is a mapping $\sigma : E \rightarrow \{+, -\}$ on the edges. It is also required that the label of a loose edge is $+$ and that of a half-edge is $-$.

An edge of a signed graph with two distinct endnodes is called a *link*. A *path* is a sequence of links e_1, \dots, e_k where e_i and e_{i+1} ($i = 1, \dots, k-1$) have a common endnode, but none of these nodes is repeated. If e_1 and e_k also have a common endnode for $k \geq 3$ (and two common endnodes for $k = 2$), then the path is *closed*. A closed path, a loop or a half-edge is called a *cycle*. A cycle is called *positive* (*negative*), if the product of the signs of its edges is positive (negative).

Clearly, a bidirected graph is a signed graph, bidirected edges are negative, directed and loose edges are positive. Conversely, the edges of a signed graph Σ can be *oriented* to get a bidirected graph, denoted $\vec{\Sigma}$. To do so, we allocate arbitrary signs to the ends of every edge so that positive edges become directed and negative edges become bidirected. More formally, if u and v are (possibly coinciding) endnodes of a link or loop e and the sign of e at u is $s(e, u)$, then the sign of e at v is $s(e, v) = -\sigma(e)s(e, u)$.

Basic operations on signed graphs, such as deletion, contraction and switching are defined in the same way as for bidirected graphs (see Section 3.2). A subgraph of a signed graph achieved by deletions and contractions of edges is sometimes called the *minor* of the graph. Two signed graphs that can be obtained from each other by switchings are called *switching equivalent*.

Now we derive the node-edge incidence matrix of a signed graph, it is the node-edge incidence matrix of the bidirected graph obtained by an orientation. Different orientations yield different node-edge incidence matrices. That is, the incidence matrix of a signed graph is not unambiguously defined, but any incidence matrix conveys all the information about the signed graph. In fact, two bidirected graphs can be oriented versions of two switching equivalent signed graphs if and only if one can be obtained from the other by switchings and edge direction reversals.

4.6.2 Matroids

A *matroid* M is a finite ground set S and a collection ϕ of subsets of S such that (I1)-(I3) are satisfied.

- (I1) $\emptyset \in \phi$.
- (I2) If $X \in \phi$ and $Y \subseteq X$ then $Y \in \phi$.
- (I3) If X, Y are members of ϕ with $|X| = |Y| + 1$ there exists $x \in X \setminus Y$ such that $Y \cup \{x\} \in \phi$.

Subsets of S in ϕ are called the *independent sets*, a maximal independent subset in S is a *basis* of M . The *rank* of M , which is the cardinality of a basis, is called the rank of the matroid, denoted as $r(M)$. A *circuit* is a minimal dependent subset of S . If B is a basis of M and $s \in S \setminus B$, then there is a unique circuit in $B \cup \{s\}$, called the *fundamental circuit of s with respect to B* . There are several different but equivalent ways to define a matroid. For example, one can define a matroid on a given ground set through its bases, rank-function, circuits, or fundamental circuits. A matroid M is called *connected* if for every pair of distinct elements x and y of S there is a circuit of M containing x and y . Otherwise, it is disconnected.

An important class of matroids is that of the *uniform matroids*. If $|S| = m$, and ϕ contains all the subsets with at most r elements ($r \leq m$), then (S, ϕ) is the uniform matroid of rank r on m elements, denoted as U_m^r .

An other standard example of matroids is when S is a finite set of vectors from a vector space over a field \mathbb{F} and ϕ contains the linearly independent subsets of S . This kind of matroid is called the *linear matroid*. If for a matroid M there exists a field \mathbb{F} such that M is a linear matroid over \mathbb{F} , then M is *representable* over \mathbb{F} . Matroids representable over $GF(2)$, the field with two elements, are called *binary*, and matroids representable over $GF(3)$, the field with three elements, are called *ternary*. A matroid representable over every field is *regular*. For a matroid to be regular, it must obviously be binary and ternary. It turns out that this condition is also sufficient.

Theorem 4.19 (Tutte [54, 56]) *A matroid is regular if and only if it is $GF(2)$ - and $GF(3)$ -representable.*

Linear matroids can be represented by matrices. Let M be a linear matroid on a finite set of vectors over \mathbb{F} . The matrix In made up of these vectors is a *standard representation matrix* of M . There is a one-to-one correspondence between linearly independent columns of In and independent sets in M , so the linear matroid $M = M(In)$ can be fully given by its representation matrix In . Obviously, deleting linearly dependent (over \mathbb{F}) rows from In does not have any effect on the structure of independent columns. Therefore, we can assume that the rows of A are linearly independent.

If In is the node-edge incidence matrix of a (connected) undirected graph G , then the corresponding binary matroid $M(In)$ is called *graphic*. A graphic matroid can be viewed as defined on the edges of the graph, the independent sets are the (edge-sets of) forests, a circuit is a cycle, a basis is a spanning tree. It is known (see [58]) that for a given connected graph G with no loop and at least three vertices, the graphic matroid $In(G)$ is connected if and only if G is 2-connected.

There is another, more compact representation matrix of a linear matroid $M(In)$. To get it, first delete linearly dependent (over \mathbb{F}) rows from In , if there are any, then choose a basis B of M . It corresponds to a basis of In , also denoted by B . By multiplying In by the inverse of B (computed over the field \mathbb{F}), the submatrix B can be converted to an identity matrix. It is clear that the independence is not affected by this operation, so the transformed matrix

also represents M . This remains true if the full-row-size identity submatrix is deleted. The remaining matrix, denoted as A , is a *compact representation matrix of M over \mathbb{F}* . The rows (respectively, columns) of A correspond to the vectors in (respectively, out of) the basis B . Note that a matroid might have several different compact representation matrices, depending on the selection of basis B . A column s of A gives us the fundamental circuit of s with respect to the chosen basis B . In fact, the basic vectors whose rows contain non-zeros in column s are the basic elements of the fundamental circuit of s . As an example, take the following matrix over $GF(3)$.

$$A = \begin{array}{c|cc} & s_3 & s_4 \\ \hline s_1 & 1 & 1 \\ \hline s_2 & 1 & -1 \end{array} \quad (4.5)$$

The ternary matroid represented by A is the uniform matroid U_4^2 . Its ground set has four vectors $\{s_1, s_2, s_3, s_4\}$. Subset $B = \{s_1, s_2\}$ is a basis. The fundamental circuit of say s_3 is $\{s_1, s_2, s_3\}$.

Graphic matroids provide an other example. Take a connected undirected graph G and its node-edge incidence matrix In , which is the standard representation of the graphic matroid based on G . First we delete a row to make In a full row rank matrix In' . Selecting a basis B of In' , which correspond to a spanning tree of G , and pivoting on its elements is equivalent to premultiplying In' by the inverse of B (all the operations are done modulo 2). As a result, we get a matrix A the columns of which give the fundamental cycles of G with respect to B , i.e., the unique cycle that contains exactly one non-tree edge. In other words, A is an unsigned (i.e., modulo 2) network matrix (see Section 2.3).

Lemma 4.20 *Any compact representation matrix of a graphic matroid can be signed with $\{+, -\}$ to obtain a network matrix. Conversely, the binary support of any network matrix is the compact representation matrix of a graphic matroid.*

Basic operations on matroids are dualization, deletion and contraction. If $M = (S, \phi)$ is a matroid, then $M^* = (S, \{S \setminus X : X \in \phi\})$ is also a matroid, called the *dual* of M . If M is linear and A is a compact representation matrix of M , then A^T is a compact representation matrix of M^* . A consequence of this fact is that if M is representable over a field \mathbb{F} , then so is the dual of M . In matroid terminology, dualization is usually expressed by the 'co' prefix. Thus, if M^* is graphic, then M is *cographic*. We state one of the most beautiful theorems in combinatorial theory characterizing the graphs embeddable in the plane by use of their co-graphic matroid (see [58]).

Theorem 4.21 *Let G be an undirected graph. The co-graphic matroid $M^*(G)$ is graphic if and only if G is planar.*

The *deletion* of $X \subseteq S$ from M results in a matroid (denoted as $M \setminus X$) on $S \setminus X$ the independent sets of which are in $\{Y \subseteq S \setminus X : Y \in \phi\}$. The matroid resulting from the *contraction* of a set $X \subseteq S$ in M is defined as $M/X = (M^* \setminus X)^*$. Deletion and contraction in graphic matroids are naturally expressed by deletion and contraction of edges. A matroid achieved by contractions and deletions in M is called a *minor* of M . For any minor N of

a linear matroid M one can find a compact representation matrix A of M such that N is represented by a submatrix of A . As a corollary, representability over a field is maintained under minor-taking. It is a classical result that binary matroid can be characterized by forbidden uniform minors.

Theorem 4.22 (*Tutte [54]*) *A matroid is binary if and only if it does not have U_4^2 minors.*

Another kind of characterization is when matroids are decomposed to simpler matroids in special ways. The most striking of this kind of results is the decomposition of regular matroid, due to Seymour [46]. It claims that the building blocks of a regular matroid are graphic matroids, cographic matroids, or matroids represented by the following compact representation matrix.

$$A(R_{10}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The special way regular matroids are built up are through 1-sums, 2-sums and 3-sums, which we do not define here. The interested reader can find the definitions in e.g., [53]. The decomposition theorem of regular matroids goes as follows.

Theorem 4.23 *Every regular matroid can be produced from graphic and cographic matroids and R_{10} by consecutive 1-, 2-, and 3-sums. Conversely, every matroid produced this way is regular.*

The importance of regular matroids is very much connected to the following theorem.

Theorem 4.24 (*Tutte [54, 56]*) *A matroid is regular if and only if it has a binary compact representation matrix the 1s of which can be replaced by ± 1 so that the resulting real matrix is totally unimodular.*

Theorem 2.6 in Section 2.3 is a consequence of Theorems 4.23 and 4.24.

4.6.3 The signed-graphic matroid

In Lemma 4.20 we stated the well-known result that unsigned network matrices are compact representation matrices over $GF(2)$ of graphic matroids. In graphical terms this means that with any direction of the edges of G , the undirected graph underlying the graphical matroids, leads to a network matrix defined on the now directed graph G . We see an analogous result with binet matrices.

Let Σ be a signed graph. The *signed-graphic matroid* of Σ is denoted by $M(\Sigma)$. The element set of $M(\Sigma)$ is $E(\Sigma)$ and a circuit of $M(\Sigma)$ falls in one of the following categories.

- (i) it is a loose edge, or
- (ii) a positive cycle, or

- (iii) a pair of negative cycles with exactly one common node, or
- (iv) a pair of disjoint negative cycles along with a minimal connecting path.

Now take a bidirected graph G , remove the orientation of the edges to get a signed graph Σ . By Corollary 3.6, the linear matroid of the node-edge incidence matrix of G is the signed-graphic matroid of Σ . Thus, any binet matrix based on the bidirected graph G is the compact representation matrix (over \mathbb{R}) of the signed-graphic matroid of Σ , as it is obtained from the node-edge incidence matrix by \mathbb{R} -pivots.

Theorem 4.25 *If $M(\Sigma)$ is a signed-graphic matroid based on signed graph Σ , and $\vec{\Sigma}$ is obtained from Σ by orienting the edges, then any binet representation based on $\vec{\Sigma}$ is a compact representation matrix of $M(\Sigma)$ over \mathbb{R} .*

We know from Lemma 4.3 that switchings in a bidirected graph do not alter its binet matrices. This phenomenon can be expressed in matroidal terms too. Notably, if two signed graphs are switching equivalent, then their signed-graphic matroids are the same.

As binet matrices generalize network matrices, the class of signed-graphic matroids contains all graphic matroids. To be even more specific, if all cycles in a signed graph Σ are positive, then $M(\Sigma)$ is the graphic matroid of Σ . At the other end of the scale, the signed-graphic matroid of an undirected graph, which is a signed graph where all the edges are negative, is the even-cycle matroid, employed by Doob [18].

It is shown in [62] that minors of the signed-graphic matroid of Σ correspond to the minors of Σ . This is equivalent to saying that deletions and contractions of edges in a signed graph correspond to deletions and contractions in its signed-graphic matroid.

Theorem 4.26 *The class of signed-graphic matroid is closed under minor-taking.*

It is a standard technique of matroid theory to find *minimal violators* for a given property, i.e., matroids that do not have this property but all their minors do. Zaslavsky gave some minimal violators of signed-graphic matroids.

Theorem 4.27 (Zaslavsky [61]) *U_4^2 is a signed-graphic matroid. U_5^2 is not a signed-graphic matroid.*

Proof. The matroid U_4^2 is signed-graphic since the compact representation matrix A in (4.5) is a binet representation of the following bidirected graph, where every pair of edges corresponds to a basis of U_4^2 .

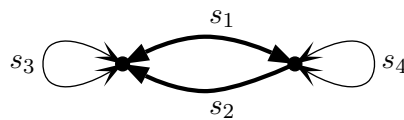


Figure 4.5: A binet representation of the matrix A given in (4.5).

To show that U_5^2 is not signed-graphic, we eliminate all the cases. First, if a signed graph Σ has the signed-graphic matroid U_5^2 , then it must have 5 edges and any two of them form a basis. In other words, any subgraph with at least three edges is not independent, but all subgraphs with two edges are such. This rules out signed graphs on a single node, as they do not have two independent edges. Let us assume that there are no isolated nodes in Σ . If a signed graph has more than one component, then the union of independent subgraphs of the components is also independent. It follows that Σ cannot have more than two components. If it has two components, then at least one of them has at least two edges. Taking two edges forming a basis from this component and one edge from the other component would result in an independent set with three edges, a contradiction.

Thus, Σ is connected. There cannot be three parallel edges in Σ because they would form a circuit, which is impossible in signed graphs by definition. If Σ has more than three nodes, then there would be a tree with three edges in it, i.e., an independent set with more than two edges. For a similar reason, there cannot be a half-edge or a loop in Σ if it has three nodes. If Σ has three nodes, the only possible structure left is a triangle in which two edges are repeated so that the graph contains two cycles of length 2. Both cycles of length 2 must be negative to form a basis, so adding an extra edge to either of them would form an independent set with more than two edges.

We are left with the case when Σ has exactly two nodes. It can not have more than two of half-edges and loops because then two of them would be incident to the same node and they would not be independent. But then Σ must contain three parallel links, which we already ruled out. We examined all the possible cases and they all led to a contradiction, so U_5^2 is not signed-graphic. ■

A very recent result due to Daniel Slilaty [50] and independently by Hongxun Qin and Thomas A. Dowling gives a technique to find minimal forbidden minors for signed-graphic matroids. They proved that a connected cographic matroid $M^*(G)$ (i.e., a matroid that is not a 1-sum of two smaller matroids, and its dual is the graphic matroid based on the undirected graph G) is signed-graphic, if and only if G can be embedded into the projective plane (see also [2]). Thus connected cographic matroids of minimally non-embeddable graphs are all minimal forbidden minors for signed-graphic matroids. This is an extension of Theorem 4.21.

Theorem 4.28 (Slilaty [50]) *A connected and cographic matroid $M^*(G)$ is signed-graphic if and only if G is a 2-connected projective-planar graph (save perhaps for some isolated vertices).*

A reformulation of Theorem 4.28 in terms of matrices is as follows.

Theorem 4.29 *Let G be a 2-connected graph and T a spanning tree. Let \vec{G} be a digraph obtained from G by orienting the edges, and A the network matrix with respect to \vec{G} and $\vec{T} \subseteq \vec{G}$. Then A^T is a binet matrix if and only if G is embeddable in \mathbb{P}^2 .*

Then an algorithm for recognizing binet matrices yields a way of testing if a given graph is embeddable in \mathbb{P}^2 . However, in the literature, there exist several methods for doing this that are much more efficient than the one deriving from our recognition algorithm for binet matrices (see [38]).

Theorem 4.25 is not all that we can say about the representability of signed-graphic matroids. In fact, they are representable over any field where 2 is not a zero-divisor, which in technical terms is referred to as a field the characteristic of which is not equal to 2.

Theorem 4.30 (Zaslavsky [61]) *A signed-graphic matroid is representable over any field of characteristic not equal to 2.*

Following an idea of Michele Conforti, one can prove the following.

Theorem 4.31 *A matroid is signed-graphic if and only if it has a compact representation matrix A over $GF(3)$ so that by applying the following operations on A , the resulting matrix is binet:*

- 1) *Replacing some entries equal to 1 or 2 by -2 and -1 , respectively;*
- 2) *Multiplying some columns in A by $-\frac{1}{2}$ (working over \mathbb{R}).*

Proof. Let Σ be a signed graph, and orient the edges of Σ to get a bidirected graph $\vec{\Sigma}$. Let A be a binet matrix based on $\vec{\Sigma}$ and A' be obtained from A by multiplying the columns with nonempty $\pm\frac{1}{2}$ -support by -2 ($-2 = 1 \pmod{3}$). Since a column of a binet matrix with nonempty $\pm\frac{1}{2}$ -support has no ± 2 -entry, it follows that A' is a $\{0, \pm 1, \pm 2\}$ -matrix. The signed-graphic matroid of Σ is represented by the node-edge incidence matrix In of $\vec{\Sigma}$. By Theorem 3.7, it follows that a set of column vectors in In are linearly independent over \mathbb{R} if and only if they are linearly independent over $GF(3)$. So, since A is a compact representation matrix of $M(\Sigma)$ over \mathbb{R} , A' is a compact representation matrix of $M(\Sigma)$ over $GF(3)$. This proves both parts of the theorem. ■

So it would be interesting to determine when $M(\Sigma)$ is representable over fields of characteristic two. It is shown in [59] that if a matroid M is representable over $GF(3)$, \mathbb{Q} , and a field of characteristic two, then M is representable over all fields except maybe $GF(2)$. From this result and Theorem 4.30, it follows that a signed-graphic matroid can be one of the following three types:

- (i) regular,
- (ii) representable over any field except $GF(2)$, or
- (iii) representable over any field of characteristic other than 2.

From this description of signed-graphic matroids, we deduce that $M(\Sigma)$ is binary if and only if $M(\Sigma)$ is regular. Moreover, provided that $M(\Sigma)$ is not regular, if $M(\Sigma)$ is quaternary (i.e., representable over $GF(4)$), then it is of type (ii), otherwise of type (iii). See [42] for some results about binary and quaternary signed-graphic matroids. The complete list of regular excluded minors for the class of signed-graphic matroids is given in [43] by Hongxun Qin, Daniel C. Slilaty and Xiangqian Zhou. So it remains to find when $M(\Sigma)$ is quaternary.

We saw that graphic matroids constitute the main building blocks of the class of regular matroids. Moreover, we know that signed-graphic matroids generalize graphic matroids. A particularly natural generalisation of regular matroids is the class of near-regular matroids,

that are the matroids representable over all fields except possibly $GF(2)$, while regular matroids are the ones representable over all fields. Whittle conjectures in [60] that there is a theorem for near-regular matroids similar to Theorem 4.23 that uses signed-graphic matroids and co-signed-graphic matroids as the basic terms in the decomposition.

Chapter 5

Camion bases

In what follows, we will consider a matrix as a set of column vectors. Let $M \in \mathbf{R}^{n \times m'}$ be a matrix of rank n , $\mathcal{H}(M) = \{\{x \in \mathbf{R}^n : c^T x = 0\} : c \in M\}$ and B a basis of M . $\mathcal{H}(M)$ splits up \mathbf{R}^n into a set S of full dimensional cones (regions). A *simplex region* is one which has exactly n facets. B is called a *Camion basis* if the corresponding hyperplanes determine the facets of a simplex region in S . It is known that there always exists a Camion basis. After some column permutations, we may write $M = [BN]$. If A is a matrix, we write $A \geq 0$ if each entry of A is nonnegative. It is possible to show that B is a Camion basis if and only if there exists a signing of some columns of M so that $B^{-1}N \geq 0$. Geometrically, $B^{-1}N \geq 0$ means that the column vectors of N are contained in the cone generated by B . We define $A = B^{-1}N$ and denote by $m = m' - n$ the number of columns of A .

Camion bases were first investigated by Camion [13], who proved that one always exists. The geometric counterpart was studied by Shannon [48], who gave the best lower bound of the number of simplex regions, namely twice the number of distinct hyperplanes in $\mathcal{H}(M)$. Note that this lower bound does not translate immediately to a lower bound for the Camion bases, because one Camion basis might be associated with many (and at least two) simplex regions.

There is no known polynomial-time algorithm to find a Camion basis in general. Fonlupt and Raco [24] described a finite procedure to find one based on the results of Camion. They also gave an algorithm which runs in time $O(n^3 m^2)$ for totally unimodular matrices.

In this Chapter, we present a new characterization of Camion bases, in the case where M is the column set of the node-edge incidence matrix (without one row) of a given connected digraph. Then, a general characterization of Camion bases as well as a recognition procedure which runs in $O(n^2 m')$ are given. Finally, an algorithm which finds a Camion basis is presented. For totally unimodular matrices, it is proven to run in time $O((nm)^2)$.

5.1 Camion bases of digraphs

Let $G = (V, E)$ be a connected digraph and M the $V \times E$ -incidence matrix of G . Let \tilde{M} be any submatrix of M obtained by deleting one row. Let B a basis of \tilde{M} , $T = (V, E_0)$ the corresponding spanning tree and suppose $\tilde{M} = [BN]$. We call T a *Camion tree* if B is a Camion basis. The matrix $A = B^{-1}N$ is a network matrix.

Define an auxiliary graph H associated with T as follows. The set of vertices is E_0 and

for $e, f \in E_0$, e and f are *adjacent* if and only if they share a common end-point in G and are in a same fundamental cycle. Then we have:

Proposition 5.1 *A spanning tree T is a Camion tree if and only if the auxiliary graph H is bipartite.*

Proof. By observation (2.1), a spanning tree T is a camion tree if and only if there exists an orientation of the edges of G so that for each non-basic edge $g = uv$ of G , all basic edges of the unique u - v -path in T are forward edges. Such an orientation will be called a *proper* orientation.

Suppose that there is a minimal odd cycle \mathcal{C} in H . By minimality of \mathcal{C} , the subgraph of H induced by \mathcal{C} has no chord. Using the definition of H , we deduce that the vertices of \mathcal{C} determine a star in T . Denote the central vertex of the star by v_0 . Since \mathcal{C} is an odd cycle, for each orientation of G , there are two adjacent vertices in \mathcal{C} corresponding to edges in G that are both either entering v_0 or leaving it. Thus, there is no proper orientation of the edges of T .

Now suppose that H is bipartite. We may suppose H is connected (otherwise what follows can be applied to each connected component of H). Here is a little procedure that finds a proper orientation of G .

Procedure Orientation-Propagation. Choose any element $e_0 \in E_0$ and orient it in an arbitrary way in G . Let $F = \{e_0\}$. Then successively choose an element $e \in E_0 \setminus F$ adjacent in H to some $f \in F$; put e in F and orient it so that $\{e, f\}$ determines a directed path in G .

Denote by $T(F)$ the subgraph of G whose edge set is F . Let us prove by induction on the cardinality of F that during the above procedure, F always satisfies properties a) and b) below:

- a) $T(F)$ is a tree.
- b) For all $f_1, f_2 \in F$ such that (f_1, f_2) is an edge in H , the reoriented edges f_1 and f_2 determine a directed path in G .

Let $k = |F|$. If $k = 1$, then clearly a) and b) are true. Now suppose $k \geq 1$ and let $e \in E_0 \setminus F$, $f \in F$ such that (e, f) is an edge of H . Using the definition of H and the induction hypothesis, we have that $T(F \cup \{e\})$ is a tree in G . Let $N_F(e) = \{g \in F : (e, g) \in H\}$. Since e is a hanging edge of the tree $T(F \cup \{e\})$, from the definition of H we deduce that $N_F(e) \cup \{e\}$ determines a star in G with a central node, say v_0 .

Let $f_1, f_2 \in N_F(e)$. As H is bipartite and the subgraph of H induced by F is connected by construction, there is a minimal path of even length in H between f_1 and f_2 . Since the path linking f_1 to f_2 is of minimal length, all its vertices correspond to edges of the star. So f_1 and f_2 are both entering into v_0 or leaving this node. It follows that the elements of $N_F(e)$ are all either entering into v_0 or leaving it. Thus we can orient e in such a way that $F \cup \{e\}$ satisfies b).

Finally, some elements of E_0 might not be in F . But such edges are not in any fundamental cycle and their orientation can be arbitrarily chosen. The orientation of F given by the above procedure induces a proper orientation of G . ■

Hoffman and Kruskal [35] and Heller and Hoffman [33] gave a characterization of positive network matrices. A directed graph is called *alternating* if in each circuit the edges are oriented alternately forwards and backwards. In Schrijver [44] (p. 278-279), it is shown that a $\{0, 1\}$ -matrix is a network matrix if and only if its columns are the incidence vectors of some directed paths in an alternating digraph. To prove the necessary part of the condition, the alternating digraph $G' = (E_0, F')$ is defined, where for edges $e, e' \in E_0$, (e, e') is in F' if and only if the head of e is the same as the tail of e' . In proposition 5.1, H is simply a subgraph of G' considered as an unoriented graph.

5.2 A polynomial-time recognition algorithm

We are going to see a characterization of Camion bases and a procedure that recognizes them in polynomial time. Let us remark that when one of m' or n is fixed, there is a trivial polynomial algorithm. We will present an algorithm to check for a given basis B whether B is a Camion basis in time $O(n^2 m')$.

Now suppose that B (a basis), N and $A = B^{-1}N$ are given. Consider the bipartite digraph $G(A)$, whose vertex set is the index set of the rows and columns of A and $e = (i, j)$ is an edge of $G(A)$ if and only if $a_{ij} \neq 0$. Set a weight function w on the set of edges:

$$w((i, j)) = \begin{cases} 2 & \text{if } \text{sign}(a_{ij}) = +, \\ 0 & \text{if } \text{sign}(a_{ij}) = -. \end{cases}$$

If C is a cycle of $G(A)$, set $w(C) = \sum_{e \in C} w(e)$. The weight function on the edges of the graph $G(A')$ will be denoted by w' . Then, we have the following characterization of Camion bases.

Proposition 5.2 *A basis B is a Camion basis if and only if each non-oriented cycle of $G(A)$ has a total weight equal to $0 \pmod{4}$.*

Lemma 5.3 *Assume that A' is obtained from A by successive applications of signing operations. If C is a cycle of $G(A)$, $w'(C) \equiv w(C) \pmod{4}$.*

Proof. We can assume that A' is obtained from A by application of a signing operation. Note that $G(A) = G(A')$. We can assume that C is an elementary cycle of $G(A) = G(A')$. But clearly $w'(C) = w(C) \pm 4$ or $w'(C) = w(C)$ and the result follows. ■

Proof of Proposition 5.2. First we prove the necessity. Let B be a Camion basis. There exists a matrix $A' \geq 0$ which can be obtained from A by successive applications of signing operations. If C is a cycle of $G(A') = G(A)$, $w'(C) = 2|C|$ and $w'(C) \equiv 0 \pmod{4}$ since C is even. The result follows from the previous lemma.

To prove the sufficiency part, we can assume that $G(A)$ is connected and we use the procedure below. In particular, the procedure returns a cycle with a total weight equal to $2 \pmod{4}$ if B is not a Camion basis. For a given spanning tree T of $G(A)$ and an edge $e \in G(A) - T$, let C_e denote the unique cycle in $T \cup \{e\}$.

Procedure IS_CAMION(A)**Input:** A matrix $A \in \mathbf{R}^{n \times m}$, where $A = B^{-1}N$ such that $G(A)$ is connected.**Output:** Yes (if B is a Camion basis) with a matrix $A' \geq 0$ or No (otherwise) with a certificate, that is a cycle in $G(A')$ of total weight equal to $2 \pmod{4}$, where A' is A after some signing operations.

- 1) Let T be a subset of edges of $G(A)$ which induce a spanning tree of $G(A)$.
Assign to an initial node v of $G(A)$ the label $l(v) = +1$, the other nodes are unlabelled.
- 2) **while** $\exists (i, j) \in T$ with i (resp. j) labelled and j (resp. i) unlabelled **do**
 label j (resp. i) with the label $+1$ or -1 in such a way that $a_{ij} \cdot l(i) \cdot l(j) > 0$.
- 3) Let A' be the matrix obtained by multiplying each row i of A by its label $l(i)$ and each column j of A by its label $l(j)$. If $A' \geq 0$, output Yes and A' .
Otherwise, there exists $e = (i, j) \notin T$ such that $a'_{ij} < 0$. Output No and C_e .

All the nodes of $G(A)$ are labelled by $+1$ or -1 at the end of this procedure. Let A' be the matrix constructed at step 3). Note that for all $(i, j) \in T$, $a'_{ij} = a_{ij} \cdot l(i) \cdot l(j) > 0$.

If B is not a Camion basis, there exists $e = (i, j) \notin T$ such that $a'_{ij} < 0$ and $w'(e) = 0$. We have $w'(C_e) \equiv 2 \pmod{4}$ since $w'(C_e) = 2(|C_e| - 1)$ and C_e is even. But by the previous lemma we have also $w(C_e) \equiv 2 \pmod{4}$.

If B is Camion, then for each edge $e = (i, j) \in G(A) - T$, $w'(C_e) \equiv 0 \pmod{4}$ as proved above and since $w'(C_e) = 2(|C_e| - 1) + w'(e)$, it follows that $w'(e) = 2$ and $a'_{ij} > 0$. ■

Corollary 5.4 *There is a polynomial-time algorithm to check whether a basis B is Camion.*

Proof. To check whether a given basis B is a Camion basis, we simply apply the procedure IS_CAMION to the matrix $A = B^{-1}N$. Since the complexity of calculating $B^{-1}N$ is $O(n^2m')$, computing A dominates the procedure IS_CAMION. Thus, the complexity of checking a given basis B is $O(n^2m')$. ■

5.3 Searching for a Camion basis

Suppose that B , N (and $A = B^{-1}N$) are given. A switching operation corresponds to the replacement of a basic vector by a nonbasic one. Remark that a pivoting operation is described in general on the standard matrix $M = [I, A]$. For our purpose, it is convenient to use the "condensed" matrix by ignoring the identity matrix part. Finding a Camion basis is equivalent to transforming the matrix A into a nonnegative one by application of the following two operations:

- pivoting operations.
- signing operations.

Signing a row of A is equivalent to signing an associated basic vector. A matrix A' obtained from A after some signing and pivoting operations will be called *equivalent* to A . Denote by $\epsilon(A)$ the set of matrices that are equivalent to A . Since the number of bases of A

and possibilities of signing some columns of A is finite, the cardinality of $\epsilon(A)$ is finite. Let us remark that for an implementation of the algorithm that is described below, it is important to maintain the correspondence between the rows of A and the basic vectors and between the columns of A and the nonbasic ones.

We will see an algorithm, called *Simp*, that runs in $O(\Delta^3(nm)^2)$ if M is an integral matrix, where Δ is the greatest determinant (in absolute value) of a basis. So, for the particular case of totally unimodular matrices (where $\Delta = 1$), *Simp* is faster than the algorithm of Fonlupt and Raco [24]. Moreover, the procedure *Simp* applied to real matrices is also finite.

For $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, let us denote by A_i the i th row of A and $A_{\bullet j}$ the j th column of A . Moreover, $A^{(i,j)}$ denotes the matrix obtained from A by pivoting on a_{ij} . For any vector or matrix V , the function $\text{sum}(V)$ evaluates the sum of all components of V .

Let us present the algorithm *Simp*.

Procedure: *Simp*(A)

Input: A real matrix A of size $n \times m$.

Output: A nonnegative matrix A' (obtained from A by a sequence of pivots and signings).

- 1) **for** $i = 1, \dots, n$ **do**
 if $\text{sum}(A_i) < 0$, **then** multiply the i th row by -1 ;
- 2) **for** $j = 1, \dots, m$ **do**
 if $A_{\bullet j} \leq 0$, **then** multiply the j th column by -1 ;
- 3) **if** $A \geq 0$, **return** A ;
 otherwise, let k, k' and l such that $a_{kl} > 0$ and $a_{k'l} < 0$; **do**
 if $\text{sum}(A_{\bullet l}) < 1$, **then** pivot on a_{kl} ;
 otherwise pivot on $a_{k'l}$;
 go to step 1;

Theorem 5.5 *For any real matrix A , the procedure *Simp* returns a nonnegative matrix after a finite number of steps. Furthermore if M is integral, then *Simp* runs in time $O(\Delta^3(nm)^2)$.*

Proof. Suppose that $\text{sum}(A_i) < 0$ at step 1 or $A_{\bullet j} \leq 0$ at step 2 for some i or some j . Let A' be the matrix obtained from A by multiplying its i th row by -1 (at step 1) or its j th column by -1 (at step 2). Clearly, we have $\text{sum}(A') > \text{sum}(A)$.

At the beginning of step 3, $\text{sum}(A_i) \geq 0$ for all i and there is no column $A_{\bullet j}$ such that $A_{\bullet j} \leq 0$. So, if there exists an entry $a_{k'l} < 0$, then there exists k such that $a_{kl} > 0$. Denote by \mathbf{c} the i th row of A without the l th column and \mathbf{b} the l th column of A without the i th row. We have that

$$A^{(i,l)} - A = \begin{pmatrix} \frac{1}{\alpha} - \alpha & \frac{\mathbf{c}}{\alpha} - \mathbf{c} \\ -\frac{\mathbf{b}}{\alpha} - \mathbf{b} & -\frac{\mathbf{bc}}{\alpha} \end{pmatrix} = \frac{1}{\alpha} \begin{pmatrix} 1 - \alpha \\ -\mathbf{b} \end{pmatrix} \cdot (1 + \alpha \quad \mathbf{c}).$$

(Note that in the middle of this equation, the matrix is written as if $i = l = 1$.) Thus,

$$\text{sum}(A^{(i,l)}) - \text{sum}(A) = \frac{1 - \text{sum}(A_{\bullet l})}{a_{il}} (\text{sum}(A_i) + 1) \text{ for } 1 \leq i \leq n.$$

So, if $\text{sum}(A_{\bullet l}) \neq 1$, $\text{sum}(A)$ will increase by pivoting either on a_{kl} or $a_{k'l}$. If $\text{sum}(A_{\bullet l}) = 1$, then $\text{sum}(A)$ neither increases nor decreases. However, since $\sum_{j \neq l} a_{k'j} \geq 0$ and $a_{k'l} < 0$, we deduce that $\text{sum}(A_{k',l}^{(k',l)}) = \frac{1}{a_{k'l}}(1 + \sum_{j \neq l} a_{k'j}) < 0$. Therefore, $\text{sum}(A)$ will increase at the next step 1.

As $|\epsilon(A)| < \infty$, it follows that Simp generates a well-oriented matrix after a finite number of steps.

Now, suppose that M is integral. By Cramer's rule, for each matrix A' equivalent to A , we have $a'_{ij} = \pm \frac{\det(B'')}{\det(B')} \forall i, j$, where B', B'' are two bases of M . Since $\det(B'') \leq \Delta$ and $|\det(B')| \geq 1$, we deduce that $a'_{ij} \leq \Delta \forall i, j$. Moreover, if $A' = (B')^{-1}N'$ where B' is a basis and N' a submatrix of M after some signing operations of columns of M , then we may write each entry of A' as a fraction of an integer over $\det(B')$. Thus $\text{sum}(A')$, respectively $\text{sum}(A)$, is a ratio of some integer to $\det(B')$, respectively $\det(B)$. So, if A' is obtained from A at some step and $\text{sum}(A') - \text{sum}(A) > 0$, then $\text{sum}(A') - \text{sum}(A) \geq \frac{1}{\Delta^2}$. Since $\text{sum}(A)$ never decreases, but increases after at most two passages at step 1 and is between $-\Delta nm$ and Δnm , the number of passages at step 1 is $O(\Delta^3 nm)$.

Since the number of elementary operations at steps 1, 2 and 3 is $O(nm)$, we conclude that the complexity of Simp is $O(\Delta^3(nm)^2)$. ■

For dealing with matrices having entries in $\{0, \pm 1, \pm \frac{1}{2}, \pm 2\}$, for instance binet matrices, we easily deduce the following procedure.

Procedure: Camion(A)

Input: A real connected matrix A of size $n \times m$ with no two identical columns.

Output: Either a connected matrix $A' \in \{0, \frac{1}{2}, 1, 2\}^{n \times m}$ such that A' is binet and

$m \leq 4 \binom{n}{2} + 2n + 1$ if and only if the input matrix A is binet, or determines that A is not binet.

- 1) if $m > 4 \binom{n}{2} + 2n + 1$, then STOP: output that A is not binet;
- 2) check the elements of A ; if it has an element other than $0, \pm 1, \pm \frac{1}{2}$, or ± 2 , then STOP: output that A is not binet;
- 3) **for** $i = 1, \dots, n$ **do**
 if $\text{sum}(A_i) < 0$, **then** multiply the i th row by -1 ;
- 4) **for** $j = 1, \dots, m$ **do**
 if $A_{\bullet j} \leq 0$, **then** multiply the j th column by -1 ;
- 5) **if** $A \geq 0$, **return** $A' = A$;
 otherwise, let k, k' and l such that $a_{kl} > 0$ and $a_{k'l} < 0$; **do**
 if $\text{sum}(A_{\bullet l}) < 1$, **then** pivot on a_{kl} ;
 otherwise pivot on $a_{k'l}$;
 go to step 2;

Theorem 5.6 *The output of procedure Camion is correct. The running time of Camion is $O((nm)^2)$.*

Proof. If A is binet, then by Lemma 4.10 $m \leq 4 \binom{n}{2} + 2n + 1$, and Camion does not stop in step 1. By Lemmas 4.4, 4.7 and 4.8, the class of connected binet matrices is closed under pivoting and signing operations. Moreover, by Lemma 4.1, any entry of a binet matrix is equal to 0, $\pm\frac{1}{2}$, ± 1 , or ± 2 . So, the number of passages through step 5 is clearly bounded by $3nm + 1$, because of step 2 and $\text{sum}(A)$ increases either at step 3 or 5 (see the proof of Theorem 5.5). This concludes the proof of Theorem 5.6. ■

Chapter 6

Recognizing binet matrices

In this chapter, we turn to the problem of recognizing binet matrices. This problem is in the complexity class \mathcal{NP} , because it is easy to verify that a matrix is binet, one only has to give a binet representation. Appa and Kotnyek formulated it as a mixed integer programming (MIP) problem (see [36]). Unfortunately, their method is not polynomial. We present here a combinatorial polynomial-time algorithm, called Binet. Throughout this chapter, using Lemmas 4.4 and 4.6, we will assume that any input matrix for our recognition problem does not have two identical columns and is connected. We shall prove the following main result.

Theorem 6.1 *A rational matrix of size $n \times m$ can be tested for being binet in time $O(n^6m)$ using the algorithm Binet.*

For network matrices, the parallel question is answered and discussed in Section 2.3. The approach of Schrijver's method outlined on page 31 (see [44]) cannot be directly adapted to binet matrices because the basis of a binet graph, a forest of negative 1-trees is more complex than a tree. Nevertheless, in the case where A is an integral and cyclic matrix, the algorithm Binet can be viewed as a generalization of Schrijver's method.

Section 6.1 is devoted to the description of the algorithm Binet. In Section 6.2, we provide an important subroutine of the algorithm Binet called Decomposition and a related central theorem. Other subroutines are depicted and analyzed in the next chapters.

6.1 The algorithm Binet

The algorithm Binet can be schematized by a flow chart in Figure 6.1. It takes a rational (connected) matrix A of size $n \times m$ as input and returns a binet representation $G(A)$ of A , or determines that none exists.

Let us describe the algorithm and analyze its correctness. The first task is to transform the matrix A into a nonnegative one by using pivoting and signing operations, and this is performed by the procedure Camion described in Section 5.3. By Theorem 5.6, Camion returns a connected matrix A' with entries 0, $\frac{1}{2}$, 1 or 2 and check that $m \leq 4 \binom{n}{2} + 2n + 1$, or determines that A is not binet (see Lemma 4.1); moreover, A' is binet if and only if A is binet. This first step is a key ingredient, at the center of fundamental properties in the design of the recognition algorithm.

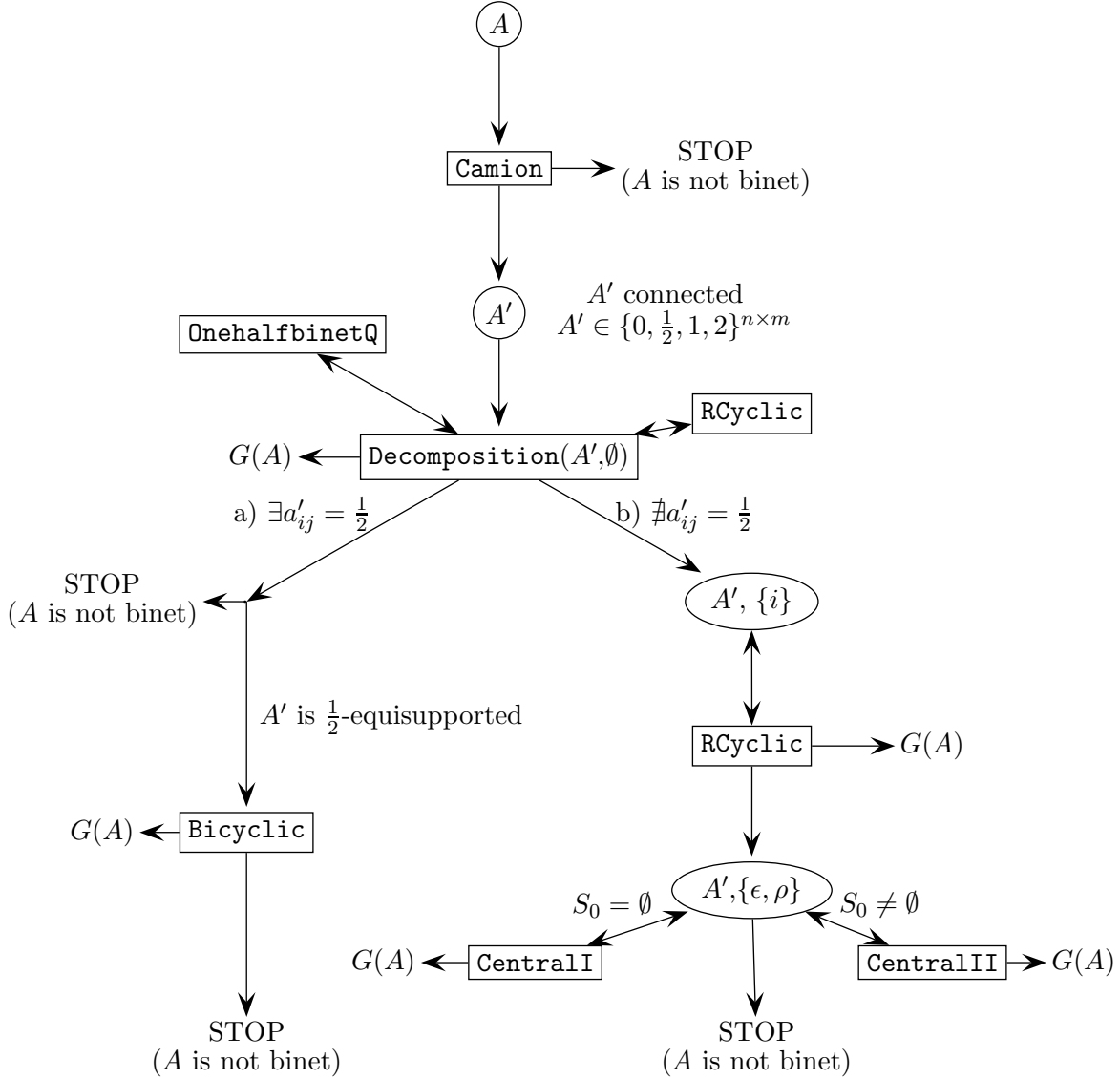


Figure 6.1: A flow chart of the algorithm Binet.

Then one makes use of the procedure Decomposition described in Subsection 6.2.2. This procedure takes as input a matrix A' and a row index subset of it denoted by Q ; it contains subroutines called RCyclic and OnehalfbinetQ that are described in Sections 8.2 and 9.2, respectively; and RCyclic is also a subroutine of OnehalfbinetQ. The procedure Decomposition is applied to matrix A' with $Q = \emptyset$. If one does not get a binet representation of A' , then by Theorem 6.2, it follows that A' is binet if and only if A' is $\frac{1}{2}$ -equisupported and bicyclic, or it is cyclic and without any $\frac{1}{2}$ -entry. Thus two cases (a and b) are distinguished.

- a) The matrix A' has a $\frac{1}{2}$ -entry.

If A' is $\frac{1}{2}$ -equisupported, then the procedure Bicyclic described in Section 10.2 is performed on A' . Let us mention here that the procedure Bicyclic uses the procedure RCyclic as subroutine. In the case where A' is not $\frac{1}{2}$ -equisupported or the procedure Bicyclic does not find any binet representation of A' , by Theorem 10.1 we conclude that

A' is not binet, and so A is not binet.

- b) The matrix A' has no $\frac{1}{2}$ -entry.

So A' is binet if and only if it is cyclic, or equivalently A' is $\{i\}$ -cyclic for some row index i or $\{\epsilon, \rho\}$ -central for some pair $\{\epsilon, \rho\}$ of row indexes. Thus one computes whether A' has an $\{i\}$ -cyclic representation for some row index i using the procedure RCyclic; if not, one checks whether A' has an $\{\epsilon, \rho\}$ -central representation for some pair $\{\epsilon, \rho\}$ of row indexes; by letting $S_0 = \{j : \epsilon, \rho \in s(A_{\bullet j})\}$, this is done using the procedure CentralI, if $S_0 = \emptyset$, or CentralII otherwise. By Theorems 8.1, 11.1 and 11.2, one obtains a cyclic representation of A' if and only if such a representation exists.

In case a or b, whenever a binet representation of A' has been found, one easily deduces a binet representation of A . Given a binet representation $G(A')$ of A' , one simply computes the inverse of all operations performed in step 1, and applies the corresponding operations on $G(A')$, in reverse order. Mathematically, the algorithm Binet is stated as follows.

Algorithm Binet(A)

Input: A (connected) matrix A of size $n \times m$.

Output: Either a binet representation $G(A)$ of A , or determines that none exists.

- 1) let $A' = A$, call **Camion**(A') of Section 5.3 by keeping in memory all pivoting and signing operations;
- 2) call **Decomposition**($A', Q = \emptyset$) of Section 6.2; if we obtain a binet representation $G(A')$ of A' , then go to 6; if A' has no $\frac{1}{2}$ -entry, then go to 4;
- 3) if A' is $\frac{1}{2}$ -equisupported, then call **Bicyclic**(A') of Section 10.2; go to 6;
- 4) **for** every row index i , **do**
 call **RCyclic**($A', \{i\}$) of Section 9.2;
 if we obtain an $\{i\}$ -cyclic representation $G(A')$ of A' , then go to 6;
endfor
- 5) **for** every pair $\{\epsilon, \rho\}$ of row indexes, **do**
 let $S_0 = \{j : \epsilon, \rho \in s(A_{\bullet j})\}$;
 if $S_0 = \emptyset$ **then**
 call **CentralI**($A', \{\epsilon, \rho\}$) of Section 11.2;
 otherwise
 call **CentralIII**($A', \{\epsilon, \rho\}$) of Section 11.3;
 endif
 if we have an $\{\epsilon, \rho\}$ -central representation $G(A')$ of A' , then go to 6;
endfor
- 6) if we have a binet representation $G(A')$ of A' , then compute in reverse order the inverse of all operations performed in step 1 on $G(A')$ and output the resulting binet representation $G(A)$ of A , otherwise output that A is not binet;

We mention here that the procedures RCyclic, CentralI and CentralIII make use of a digraph, called D , and described in chapter 7. If A has an R^* -cyclic, R^* -central or R^* -network representation $G(A)$, where R^* is a row index subset of A , then the basic forest in $G(A)$ with edge index set $\overline{R^*}$ has some nice properties that are recognizable thanks to the

digraph D . A subroutine of the procedures RCyclic and CentralI called Forest and given in Section 7.5 computes some feasible spanning forest of D .

Using the results of the next chapters, we prove here Theorem 6.1.

Proof of Theorem 6.1. The correctness of the algorithm has been proved earlier. Let us analyze its running time. By Theorem 5.6, the procedure Camion takes time $O(n^2m^2)$. Let α' be the number of nonzero entries of the matrix A' in step 2. By Theorem 6.2, the procedure Decomposition takes time $O(nm^2\alpha')$. By Theorem 10.1, the procedure Bicyclic works in time $O(nm\alpha')$. Moreover, by Theorem 8.1, the procedure RCyclic takes time $O(nm\alpha')$, and the number of passages through step 4 does not exceed n . Finally, by Theorems 11.1 and 11.2, the procedures CentralI and CentralII work in time $O(n^3\alpha')$, and the number of passages through step 5 does not exceed n^2 . If $m > 4 \binom{n}{2} + 2n + 1$, then in step 1 the subroutine Camion directly outputs that A is not binet. Otherwise, using previous arguments, we deduce that the running time of the algorithm Binet is $O(\max(n^2m^2, nm^2\alpha', n^2m\alpha', n^5\alpha'))$. Hence, since $\alpha' \leq nm$, the computational effort of the algorithm Binet is $O(n^6m)$. ■

6.2 The procedure Decomposition

Suppose we are given a connected matrix A in $\{0, \frac{1}{2}, 1, 2\}^{n \times m}$. Let α be the number of nonzero entries in A . The goal of the present section is to decompose the matrix A into smaller pieces by reducing the binet recognition problem to the simpler case of R -cyclic matrices and $\frac{1}{2}$ -binet matrices. For that purpose, we describe a procedure called Decomposition which takes the matrix A and a row index subset Q of A as input. Under certain conditions, the procedure computes a binet representation of A such that each basic edge with index in Q is a half-edge; the name of the procedure comes from the fact that it "decomposes" the matrix A into two matrices (if A has a $\frac{1}{2}$ -entry), and then works iteratively on one of both matrices. A proof of the following theorem is given.

Theorem 6.2 *The matrix A is binet if and only if one of the following three statements is valid:*

- 1) *A is bicyclic and $\frac{1}{2}$ -equisupported, or*
- 2) *A is cyclic and without any $\frac{1}{2}$ -entry, or*
- 3) *the procedure Decomposition with input A and row index subset $Q = \emptyset$ provides a binet representation of A .*

Moreover, the running time of the procedure Decomposition is $O(nm^2\alpha)$.

Before stating the procedure Decomposition and proving Theorem 6.2, we provide in Subsection 6.2.1 some intuitions and graphical ideas on which these are based in an informal way. Then, in Subsection 6.2.2, the procedure Decomposition and a formal proof of Theorem 6.2 are given.

6.2.1 An informal sketch of the procedure

Let us consider the binet matrix A given in Figure 6.2. Our aim is to construct a binet representation of A , for instance the one given in Figure 6.2, without knowing that such a representation exists. We describe some steps of the procedure Decomposition applied on A .

	f_1	f_2	f_3	f_4	f_5	f_6	f_7
e_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	0	0
e_2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0
e_3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	0	0
e_4	0	0	0	1	0	1	0
e_5	0	0	0	0	0	1	0
e_6	1	1	1	0	2	0	0
e_7	1	1	0	0	1	0	0
e_8	0	0	1	0	1	0	0
e_9	0	0	1	0	0	0	0
e_{10}	0	0	1	0	0	0	1
e_{11}	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$

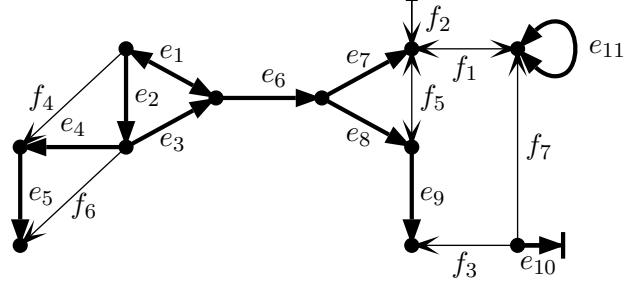


Figure 6.2: A binet matrix A with a binet representation of A .

Provided that there exists a binet representation of A , one arising question is whether one can locate the edge index set of a basic full cycle. We observe that the $\frac{1}{2}$ -support of the first and second column intersect and are not equal. Using this simple property, it will be proved that $s_{\frac{1}{2}}(A_{\bullet 1}) \cap s_{\frac{1}{2}}(A_{\bullet 2})$ is the edge index set of a full basic cycle in any binet representation of A , if one exists. Can we derive a general statement?

Once some row index subset R of A , for instance $R = \{1, 2, 3\} = s_{\frac{1}{2}}(A_{\bullet 1}) \cap s_{\frac{1}{2}}(A_{\bullet 2})$ has been located, we consider any column $A_{\bullet j}$, namely $j = 4$ and $j = 5$, whose support intersects R and such that $R \not\subseteq s_{\frac{1}{2}}(A_{\bullet j})$. Provided that A has a binet representation $G(A)$, it will be proved that the nonbasic edges f_4 and f_5 correspond to 1-edges in $G(A)$ whose fundamental circuit intersects the full cycle with edge index set R ; then, since $s(A_{\bullet 6}) \cap s(A_{\bullet 4}) \neq \emptyset$ and $R \not\subseteq s_{\frac{1}{2}}(A_{\bullet 6})$, one can show that f_6 is also a 1-edge and $R' = R \cup s(A_{\bullet 4}) \cup s(A_{\bullet 5}) \cup s(A_{\bullet 6})$ is the edge index set of a 1-tree in $G(A)$. The procedure Decomposition builds up the matrix $A(R) = A_{R' \times \{1, \dots, 6\}}$. Given a binet representation $G(A)$ of A , if one exists, one can construct an R -cyclic representation $G(A(R))$ of $A(R)$, by contracting all basic edges with index in $\overline{R'}$ and deleting the remaining loose edges as illustrated in Figure 6.3.

We may write $A = \begin{bmatrix} A(R) & O_{8 \times 1} \\ A' & \end{bmatrix}$, where A' is a submatrix of A . In general, it is not sufficient to have binet representations $G(A(R))$ and $G(A')$ of the matrices $A(R)$ and A' , respectively, to compute a binet representation of A . The procedure Decomposition constructs a matrix $A(\tau)$ whose A' (without zero columns) and τ are submatrices.

Suppose that A has a binet representation $G(A)$. Let us look at the graphical interpretation of $A(\tau)$, without giving an explicit definition of this matrix. Delete in $G(A)$ the edges in $G(A(R))$, and the remaining isolated nodes. We are left with a connected bidirected graph containing some nodes of $G(A(R))$. By adding a basic half-edge entering each left node of

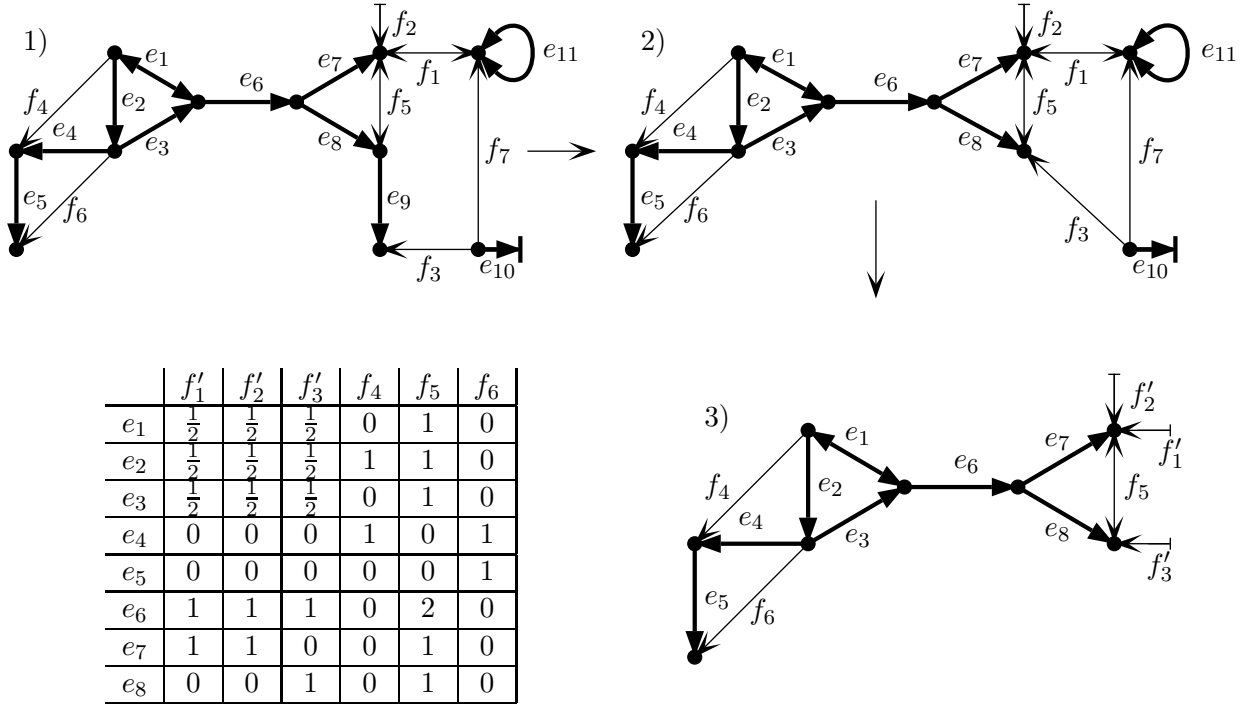


Figure 6.3: the matrix $A(R)$ obtained from A given in Figure 6.2 by the procedure Decomposition, and a $\{1, 2, 3\}$ -cyclic representation of $A(R)$ on the right of $A(R)$ obtained from $G(A)$ (from 1) to 2) edge e_9 is contracted, and from 2) to 3) edges e_{10} and e_{11} are contracted).

$G(A(R))$, we obtain a binet representation of the matrix $A(\tau)$. We have $\tau = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A(\tau)$ has $q := 2$ more rows than A' (see Figure 6.4).

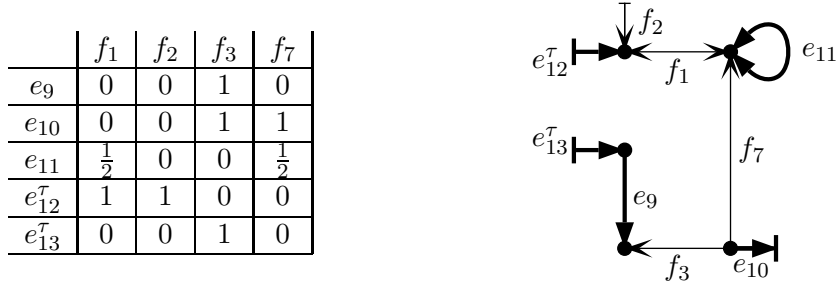


Figure 6.4: The binet matrix $A(\tau)$ and a binet representation of $A(\tau)$, where A is given in Figure 6.2.

The procedure Decomposition works iteratively by searching for a binet representation of $A(\tau)$ such that its last q rows correspond to half-edges denoted by e_{12}^τ and e_{13}^τ . If binet representations $G(A(R))$ and $G(A(\tau))$ of $A(R)$ and $A(\tau)$, respectively, have been found as in Figures 6.3 and 6.4, it is possible to construct a binet representation of A as follows. Identify the endnode of e_{12}^τ (respectively, e_{13}^τ) with the endnode of f'_1 and f'_2 (respectively, f'_3). Then

delete the edges e_{12}^τ , e_{13}^τ , f'_1 , f'_2 and f'_3 . See Figure 6.5.

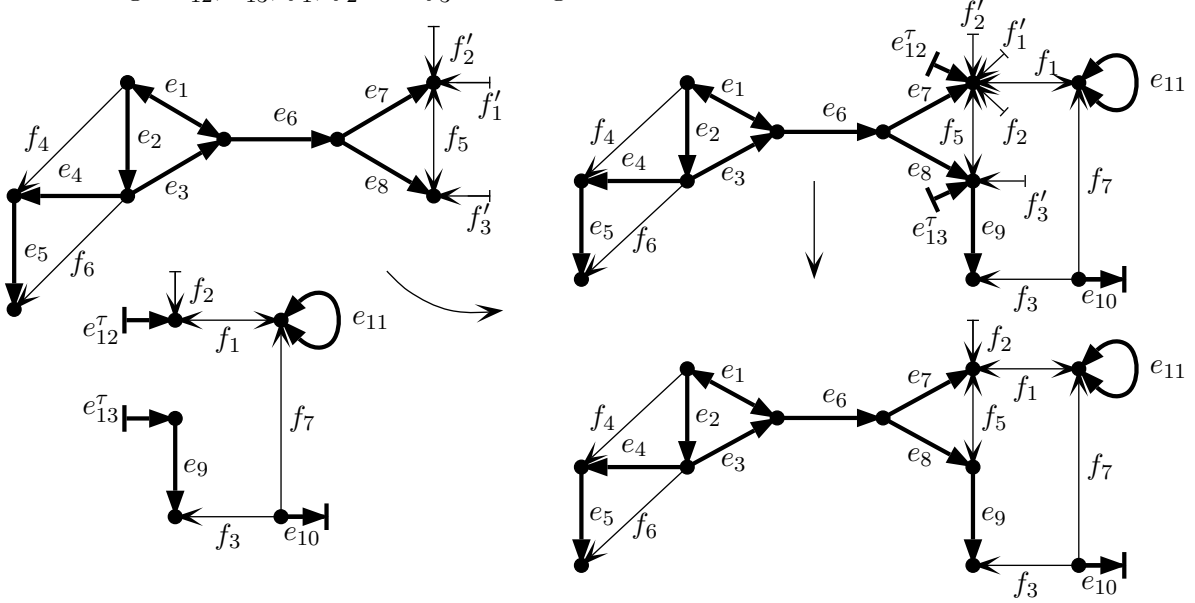


Figure 6.5: How to obtain a binet representation of A (at the bottom right) using binet representations of $A(R)$ and $A(\tau)$, where A is given in Figure 6.2.

6.2.2 The procedure Decomposition

Let us introduce the main definitions and lemmas involved in the description of the procedure Decomposition and the proof of Theorem 6.2. A pair of columns $(A_{\bullet j}, A_{\bullet j'})$ with $1 \leq j, j' \leq m$ such that $\emptyset \neq s_{\frac{1}{2}}(A_{\bullet j}) \neq s_{\frac{1}{2}}(A_{\bullet j'}) \neq \emptyset$ and $s_{\frac{1}{2}}(A_{\bullet j}) \cap s_{\frac{1}{2}}(A_{\bullet j'}) \neq \emptyset$ is called a *connective pair*.

Lemma 6.3 Suppose that A is binet and let $1 \leq j, j' \leq m$ be such that the pair $(A_{\bullet j}, A_{\bullet j'})$ is connective. Then in any binet representation of A , $s_{\frac{1}{2}}(A_{\bullet j}) \cap s_{\frac{1}{2}}(A_{\bullet j'})$ is the edge index set of a basic full cycle.

Lemma 6.4 Suppose that A is bicyclic and not $\frac{1}{2}$ -equisupported. For any column index j ($1 \leq j \leq m$) such that $s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset$, there exists j' ($1 \leq j' \leq m$) such that the pair $(A_{\bullet j}, A_{\bullet j'})$ is connective.

Lemma 6.5 Suppose that A has a binet representation $G(A)$ which is not bicyclic. For any column index j ($1 \leq j \leq m$) such that $s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset$ and there is no connective pair $(A_{\bullet j}, A_{\bullet j'})$ with $1 \leq j' \leq m$, the set $s_{\frac{1}{2}}(A_{\bullet j})$ corresponds to the edge index set of a (basic) full cycle in $G(A)$.

Proof of Lemmas 6.3, 6.4 and 6.5. Let $G(A)$ be a binet representation of A . By Lemma 4.1, an entry a_{ij} of A equal to $\frac{1}{2}$ corresponds to the weight of an edge e_i belonging to a basic full cycle in the fundamental circuit of a half-edge or 2-edge f_j . Then, by Corollary 3.6 the $\frac{1}{2}$ -support of a half-edge (respectively, a 2-edge) represents the edge index set of one basic full cycle (respectively, one or two basic full cycles). This implies Lemma 6.3.

Let j ($1 \leq j \leq m$) be a column index such that $s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset$. Let us prove Lemma 6.4. If A is bicyclic and not $\frac{1}{2}$ -equisupported, then by above arguments it follows that there exists a column index j' such that f_j and $f_{j'}$ are a 2-edge and a half-edge, respectively, or vice versa. Hence $(A_{\bullet j}, A_{\bullet j'})$ is a connective pair.

Now let us show Lemma 6.5. We assume that $G(A)$ is not bicyclic. Suppose that $s_{\frac{1}{2}}(A_{\bullet j})$ is the edge index set of two basic full cycles C_1 and C_2 . Since $G(A)$ is not bicyclic, there exists at least a third basic cycle. Thus, using the fact that A is connected, there exists a 2-edge $f_{j'}$ containing C_1 or C_2 in its fundamental circuit and a basic cycle different from these both. So the pair $(A_{\bullet j}, A_{\bullet j'})$ is connective. This proves the contrapositive of Lemma 6.5. ■

Given a row index subset R of A , the procedure Decomposition computes a row index subset R' ($\supseteq R$) and column index subsets $S_{\frac{1}{2}}$ and S_2 of A as well as a submatrix $A(R)$ by using a subroutine called MatRcyclic. Then, thanks to a subroutine called Mattau, a number q and matrices denoted by τ and $A(\tau)$ are computed, where q corresponds to the number of rows of τ . Moreover, the set $S_{\frac{1}{2}}$ is also partitioned into subsets U_1, \dots, U_q . Later, we shall prove the following: If $A(R)$ has an R -cyclic representation $G(A(R))$, then the nonbasic edges with index in $S_{\frac{1}{2}}$ are half-edges denoted by f'_j for all $j \in S_{\frac{1}{2}}$; further, for $i = 1, \dots, q$, the nonbasic edges f'_j with $j \in U_i$ have a common endnode denoted as u_i . If $A(\tau)$ has a binet representation $G(A(\tau))$, then the basic edges corresponding to the rows of τ are denoted by $e_{i_{max}+1}^\tau, \dots, e_{i_{max}+q}^\tau$, where i_{max} is the largest row index of A .

Procedure Decomposition(A, Q)

Input: A connected matrix A with entries 0, 1, 2 or $\frac{1}{2}$ and a row index subset Q of A .

Output: a binet representation $G(A)$ of A such that each element in Q is the index of a basic half-edge, or stops.

- 1) **if** A has a $\frac{1}{2}$ -entry, **then**
- 2) let j be such that $s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset$;
- 3) **if** there exists a connective pair $(A_{\bullet j}, A_{\bullet j'})$, **then**
 let $R = s_{\frac{1}{2}}(A_{\bullet j}) \cap s_{\frac{1}{2}}(A_{\bullet j'})$;
 otherwise
 let $R = s_{\frac{1}{2}}(A_{\bullet j})$;
 endif
- 4) call MatRcyclic(A, R) which outputs sets R' , $S_{\frac{1}{2}}$ and S_2 and a submatrix $A(R)$ of A ;
 then call Mattau($A, R', S_{\frac{1}{2}}, S_2$) which outputs a number q , subsets U_1, \dots, U_q of $S_{\frac{1}{2}}$
 and matrices τ and $A(\tau)$;
- 5) **if** $Q \cap R' \neq \emptyset$ or $q > |R'|$, **then** STOP: A does not have a binet representation such
 that R is the index set of a basic cycle and Q an index set of basic half-edges;
 endif ;
- 6) let i_{max} be the largest row index of A , $Q = Q \cup \{i_{max} + 1, \dots, i_{max} + q\}$;
 let $G(A(\tau)) = \text{Decomposition}(A(\tau), Q)$ and $G(A(R)) = \text{RCyclic}(A(R), R)$;
- 7) **for** $i = 1, \dots, q$, identify u_i with the endnode of $e_{i_{max}+i}^\tau$ **endfor**;
 delete f'_j for all $j \in S_{\frac{1}{2}}$ and $e_{i_{max}+i}^\tau$ for all $1 \leq i \leq q$; then output the binet
 representation $G(A)$ of A ;
 otherwise

- 8) call `OnehalfbinetQ(A,Q)` and output a binet representation of A if we have one;
endif

For a matrix A' , we define the graph $H(A')$ with respect to A' as follows. The set of vertices is the column index set of A' , and two vertices j and j' are adjacent if and only if $s(A'_{\bullet j}) \cap s(A'_{\bullet j'}) \neq \emptyset$. Let us state the subroutine `MatRcyclic`. See Figures 6.6 and 6.7 for an illustration of all sets computed by this procedure.

Procedure MatRcyclic(A,R)

Input: A matrix A and a row index subset R of A .

Output: A row index subset R' and column index subsets $S_{\frac{1}{2}}$ and S_2 of A ,
 and a submatrix $A(R)$ of A .

- 1) let $S_0 = \{j : s(A_{\bullet j}) \cap R \neq \emptyset\}$, $S_{\frac{1}{2}} = \{j : R \subseteq s_{\frac{1}{2}}(A_{\bullet j})\}$, $S_1 = S_0 \setminus S_{\frac{1}{2}}$
 and S_2 be the set of all nodes in $H(A_{\bullet \overline{S_{\frac{1}{2}}}})$ reachable from S_1 ;
 let $R' = \bigcup_{j \in S_2} s(A_{\bullet j}) \cup R$ and $A(R) = A_{R' \times (S_{\frac{1}{2}} \cup S_2)}$;
 output R' , $S_{\frac{1}{2}}$, S_2 and $A(R)$;

		S_0			S_2							
		$S_{\frac{1}{2}}$		S_1								
R'	R	$\frac{1}{2}$...	$\frac{1}{2}$	1	0	...	0	0	...	0	
		\vdots		\vdots	1	1	\vdots		\vdots		\vdots	
		$\frac{1}{2}$...	$\frac{1}{2}$		1	0	...	0	0	...	0
		1				1		2	0	...	0	
				1	2	1	$\frac{1}{2}$	1	\vdots		\vdots	
							0	...	0			
		1	1				1	...	1			
		1						1				
				1	\vdots				2			
		$\frac{1}{2}$						$\frac{1}{2}$...			
				0	...	0			

Figure 6.6: An example of a (non-binet) matrix A to illustrate the sets R' , $S_{\frac{1}{2}}$, S_0 , S_1 and S_2 , with respect to R .

Suppose that A has a binet representation $G(A)$ such that R is the edge index set of a full basic cycle, say C . Let $S_{\frac{1}{2}}$, S_0 , S_1 , S_2 and R' be the sets computed by `MatRcyclic`. The columns with index in S_0 correspond to nonbasic edges whose fundamental circuit contains edges of C . As mentioned earlier, an entry a_{ij} of A equal to $\frac{1}{2}$ corresponds to the weight of an edge e_i belonging to a basic full cycle in the fundamental circuit of a half-edge or 2-edge

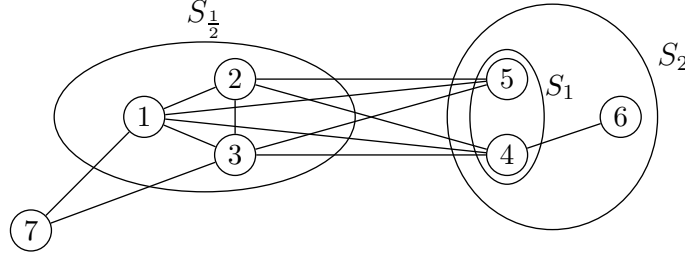


Figure 6.7: An illustration of the graph $H(A)$ and the sets $S_{\frac{1}{2}}$, S_1 and S_2 ($S_0 = S_{\frac{1}{2}} \uplus S_1$) in function of $R = \{1, 2, 3\}$ and A given in Figure 6.2 ($R' = \{1, 2, 3, 4, 5, 6, 7, 8\}$).

f_j . Moreover, the $\frac{1}{2}$ -support of a half-edge (respectively, a 2-edge) represents the edge index set of one basic full cycle (respectively, one or two basic full cycles). Thus S_1 is the subset of S_0 of 1-edge (except half-edge) indexes, and $S_{\frac{1}{2}}$ is the index set of nonbasic half-edges and 2-edges whose fundamental circuit contains C . The set S_2 is an index set of 1-edges (without any half-edge). So the $\frac{1}{2}$ -support of the columns with index in S_2 is empty. We observe that R' corresponds to the edge index set of a negative 1-tree in $G(A)$.

Since $H(A)$ is connected, the connected components of $H(A_{\bullet, \overline{S_{\frac{1}{2}}}})$ that do not have any vertex of S_1 are linked to at least one vertex of $S_{\frac{1}{2}}$ in $H(A)$. Moreover, $S_{\frac{1}{2}}$ induces a clique in $H(A)$ and each element of $S_{\frac{1}{2}}$ is adjacent to each vertex in S_1 .

Lemma 6.6 *If A has a binet representation such that R is the edge index set of a basic full cycle, then $A(R)$ is R -cyclic.*

Proof. Let $G(A)$ be a binet representation of A such that R is the edge index set of a basic full cycle. As mentioned earlier, the set R' computed by MatRcyclic corresponds to the edge index set of a basic negative 1-tree denoted by T . Let us contract all basic edges of $G(A)$ not in T (in particular, 2-edges with one endnode in T become half-edges). Then by deleting all left nonbasic loose edges, one obtains a binet representation $G(A(R))$ of $A(R)$ such that R corresponds to a full basic cycle (see Figure 6.3 for an example). ■

Now, let us state the subroutine Mattau.

Procedure Mattau($A, R', S_{\frac{1}{2}}, S_2$)

Input: A matrix A , a row index subset R and column index subsets $S_{\frac{1}{2}}$ and S_2 of A .

Output: a number q , subsets U_1, \dots, U_q of $S_{\frac{1}{2}}$ and matrices τ and $A(\tau)$ such that q is the number of rows of τ .

- 1) let us partition $S_{\frac{1}{2}}$ into subsets U_1, \dots, U_q so that $j, j' \in S_{\frac{1}{2}}$ are in a same subset if and only if $s(A_{\bullet, j}) \cap R' = s(A_{\bullet, j'}) \cap R'$;

- 2) let τ be the matrix given by $\tau_{ij} = \begin{cases} 1 & \text{if } j \in U_i \\ 0 & \text{otherwise,} \end{cases}$ for $i = 1, \dots, q$ and $j \in S_{\frac{1}{2}}$,

$$l = |\overline{S_{\frac{1}{2}} \cup S_2}| \text{ and } A(\tau) = \begin{pmatrix} A_{\overline{R'} \times \overline{S_2}} \\ \tau & O_{q \times l} \end{pmatrix};$$

output q, U_1, \dots, U_q, τ and $A(\tau)$;

For an example of the sets and matrices computed by Mattau, see Figure 6.4, where $U_1 = \{1, 2\}$, $U_2 = \{3\}$ and $\tau = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Notice that if $S_2 \neq \emptyset$, then the number of columns of $A(\tau)$ is strictly smaller than the number of columns of A . In the following lemmas and propositions, if we are given row index subsets R and Q of A , then we assume that the objects R' , $A(R)$, q , U_1, \dots, U_q , τ and $A(\tau)$ have been computed by the subroutines MatRcyclic and Mattau. Further, i_{\max} denotes the largest row index of A . The next lemma justifies the definition of the nodes u_1, \dots, u_q used in step 7.

Lemma 6.7 *Let R be a row index subset of A and suppose that $A(R)$ has an R -cyclic representation $G(A(R))$. Then, for $i = 1, \dots, q$, the nonbasic edges f'_j with $j \in U_i$ are half-edges having a common endnode denoted by u_i .*

Proof. Using arguments similar to those preceding Lemma 6.6, one can prove that each nonbasic edge f'_j with $j \in S_{\frac{1}{2}}$ is a half-edge. From the way of partitioning $S_{\frac{1}{2}}$ into subsets U_1, \dots, U_q and by Corollary 3.6 and Lemma 4.2, it follows that for any i ($1 \leq i \leq q$), $s(A_{\bullet j}) \cap (R' \setminus R)$ is the edge index set of a (directed) path denoted P_i for all $j \in U_i$. Notice that the initial node of P_i is a central node, and we denote by u_i the terminal node of P_i , which is the endnode of f'_j for all $j \in U_i$. ■

Lemma 6.8 *Let R and Q be row index subsets of A . If A has a binet representation such that R is the edge index set of a basic full cycle and each element in Q is the index of a basic half-edge, then $q \leq |R'|$, $Q \cap R' = \emptyset$ and $A(\tau)$ has a binet representation such that each element in $Q \cup \{i_{\max} + 1, \dots, i_{\max} + q\}$ is a basic half-edge index.*

Proof. Let $G(A)$ be a binet representation of A such that R is the edge index set of a basic full cycle, and each element in Q is the index of a basic half-edge. We have already observed that R' is the edge index set of a negative 1-tree whose basic cycle has an index set equal to R . Thus $Q \cap R' = \emptyset$.

Let G_1 be the connected subgraph of $G(A)$ formed by basic edges with index in R' and nonbasic ones indexed by S_2 . We may suppose that G_1 has a unique basic bidirected edge, and this one is entering. For each set U_i ($1 \leq i \leq q$), let P_i be the directed path in $G(A)$ with edge index set $s(A_{\bullet j}) \cap (R' - R)$ for all $j \in U_i$ (as in Lemma 6.7). Notice that the initial node of P_i is a central node, and we denote by u_i the terminal node of P_i . Clearly, $q \leq |R'|$ since $|R'|$ corresponds to the number of nodes of G_1 . Delete from $G(A)$ all edges of G_1 and then the isolated nodes. Let us call G' the remaining graph. We observe that $G_1 \cap G' = \{u_1, \dots, u_q\}$. Then, by adding a basic entering half-edge to each left node of $G_1 \cap G'$ it yields a binet representation of $A(\tau)$ such that for $i = i_{\max} + 1, \dots, i_{\max} + q$ the i th row of $A(\tau)$ corresponds to a basic half-edge incident with u_i and each element in Q is the index of a basic half-edge. ■

Proposition 6.9 *Let R and Q be row index subsets of A . Then the matrix A has a binet representation such that R is the edge index set of a basic full cycle and any element in Q is*

a basic half-edge index if and only if $q \leq |R'|$, $Q \cap R' = \emptyset$, $A(R)$ is R -cyclic and $A(\tau)$ has a binet representation such that each element in $Q \cup \{i_{\max} + 1, \dots, i_{\max} + q\}$ is a basic half-edge index.

Proof. The "only if" part has been proved in Lemmas 6.6 and 6.8.

\Leftarrow : Let $G(A(R))$ be an R -cyclic representation of $A(R)$ having exactly one basic bidirected edge (and this one is entering). Let $G(A(\tau))$ be a binet representation of $A(\tau)$ such that each element $i \in Q \cup \{i_{\max} + 1, \dots, i_{\max} + q\}$ is the index of a basic half-edge e_i^τ . By Corollary 3.6 and Lemma 4.1, it follows that the columns of $A(\tau)$ with index in $S_{\frac{1}{2}}$ correspond to half-edges or 2-edges in $G(A(\tau))$. By performing step 7 of the procedure Decomposition and using Corollary 3.6 and Lemmas 4.1 and 4.2, we obtain a binet representation of A such that R corresponds to a basic full cycle and each element in Q is a basic half-edge index (see Figure 6.5). ■

Proposition 6.10 *Given a matrix A and a row index subset $Q \neq \emptyset$ of A , if the matrix A has a binet representation such that Q is an index set of half-edges then the procedure Decomposition with input A and Q outputs such a representation.*

Proof. The proof is by induction on the number of rows of A having a $\frac{1}{2}$ -entry. Suppose that there exists a binet representation $G(A)$ of A such that Q is an index set of half-edges. If A has no $\frac{1}{2}$ -entry, then it follows that $G(A)$ is $\frac{1}{2}$ -binet (see Lemma 4.12). By Theorem 9.1, in step 8 the procedure Decomposition outputs a $\frac{1}{2}$ -binet representation of A such that Q is an index set of half-edges.

Now suppose that A has at least one $\frac{1}{2}$ -entry. Let R be the set computed in step 3. By Lemmas 6.3 and 6.5, we deduce that R is the edge index set of a full cycle in $G(A)$. By the "only if" part of Proposition 6.9, it follows that the procedure does not stop in step 5, $A(R)$ is R -cyclic and $A(\tau)$ has a binet representation such that each element in $Q \cup \{i_{\max} + 1, \dots, i_{\max} + q\}$ is a basic half-edge index. By Theorem 8.1, the subroutine RCyclic with input $A(R)$ and R outputs an R -cyclic representation of $A(R)$. Since the number of rows in $A(\tau)$ with at least one $\frac{1}{2}$ -entry is smaller than in A , by induction hypothesis, in step 6 the procedure Decomposition computes a binet representation $G(A(\tau))$ of $A(\tau)$ such that $Q \cup \{i_{\max} + 1, \dots, i_{\max} + q\}$ is an index set of basic half-edges. Using the "if part" of Proposition 6.9 (see the proof), this concludes the proof of Proposition 6.10. ■

Proof of Theorem 6.2. The "if part" is straightforward.

Let us prove the other implication. Suppose that A has a binet representation $G(A)$. Assume that if A is bicyclic then it is not $\frac{1}{2}$ -equisupported, and if A has no $\frac{1}{2}$ -entry then it is not cyclic. Let us show that the procedure Decomposition with input A and $Q = \emptyset$ outputs a binet representation of A .

If A has no $\frac{1}{2}$ -entry, then by assumption it is not cyclic and by Lemma 4.12 it follows that A is $\frac{1}{2}$ -binet. By Theorem 9.1, in step 8 the procedure Decomposition outputs a $\frac{1}{2}$ -binet representation of A .

Now suppose that A has at least one $\frac{1}{2}$ -entry. Let R be the set computed in step 3. Using Lemmas 6.3, 6.4 and 6.5, we deduce that R is the edge index set of a full cycle in $G(A)$. By the "only if" part of Proposition 6.9, it follows that the procedure does not stop

in step 5, $A(R)$ is R -cyclic and $A(\tau)$ has a binet representation such that each element in $\{i_{max} + 1, \dots, i_{max} + q\}$ is a basic half-edge index. So in step 6, by Theorem 8.1, the subroutine RCyclic with input $A(R)$ and R outputs an R -cyclic representation of $A(R)$, and by Proposition 6.10, the procedure Decomposition computes a binet representation $G(A(\tau))$ of $A(\tau)$ such that $\{i_{max} + 1, \dots, i_{max} + q\}$ is an index set of basic half-edges. Using the "if part" of Proposition 6.9 (see the proof), the procedure Decomposition outputs a binet representation of A .

Let us analyze the time needed to construct a binet representation $G(A)$ of A . One calls at most one time the procedure OnehalfbinetQ with input a matrix A' , and thanks to step 5, the number of rows of A' does not exceed n . Since the number of nonzero elements in A' is at most α , by Theorem 9.1 the subroutine OnehalfbinetQ takes time $O(nm^2\alpha)$. Denote by R_1, \dots, R_δ all row index subsets computed in step 3. The procedure Decomposition constructs an R_i -cyclic representation of some matrix $A(R_i)$, for $i = 1, \dots, \delta$. Let n_i , m_i and α_i be the number of rows, columns and nonzero elements of $A(R_i)$, respectively, for $i = 1, \dots, \delta$. Thanks to step 5 of the procedure Decomposition, we have $n_1 + n_2 + \dots + n_\delta \leq n$ and $\alpha_1 + \alpha_2 + \dots + \alpha_\delta \leq \alpha$ (and $m_i \leq m$ for all $1 \leq i \leq \delta$). By Theorem 8.1, the computational effort to obtain an R_i -cyclic representation $G(A(R_i))$ of $A(R_i)$ is $O(n_i m_i \alpha_i)$. Altogether, the time required to construct a binet representation of A is bounded by

$$C(nm^2\alpha + n_1 m_1 \alpha_1 + \dots + n_\delta m_\delta \alpha_\delta) \leq 2Cnm^2\alpha,$$

for some constant C . This completes the proof of Theorem 6.2. ■

Chapter 7

Encoding of a global structure: a digraph D

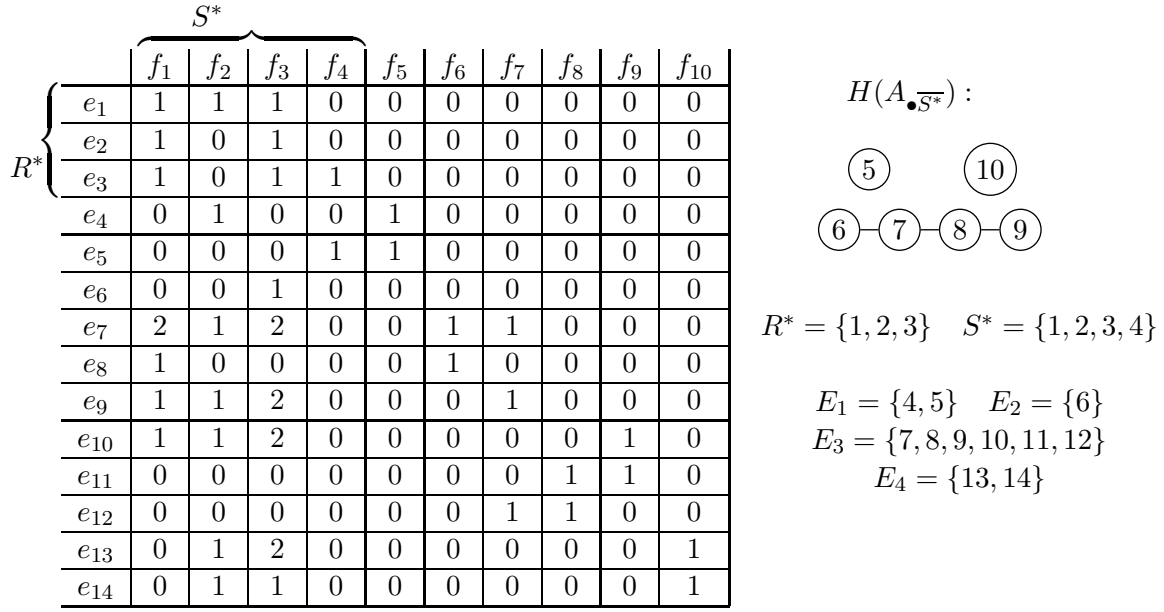
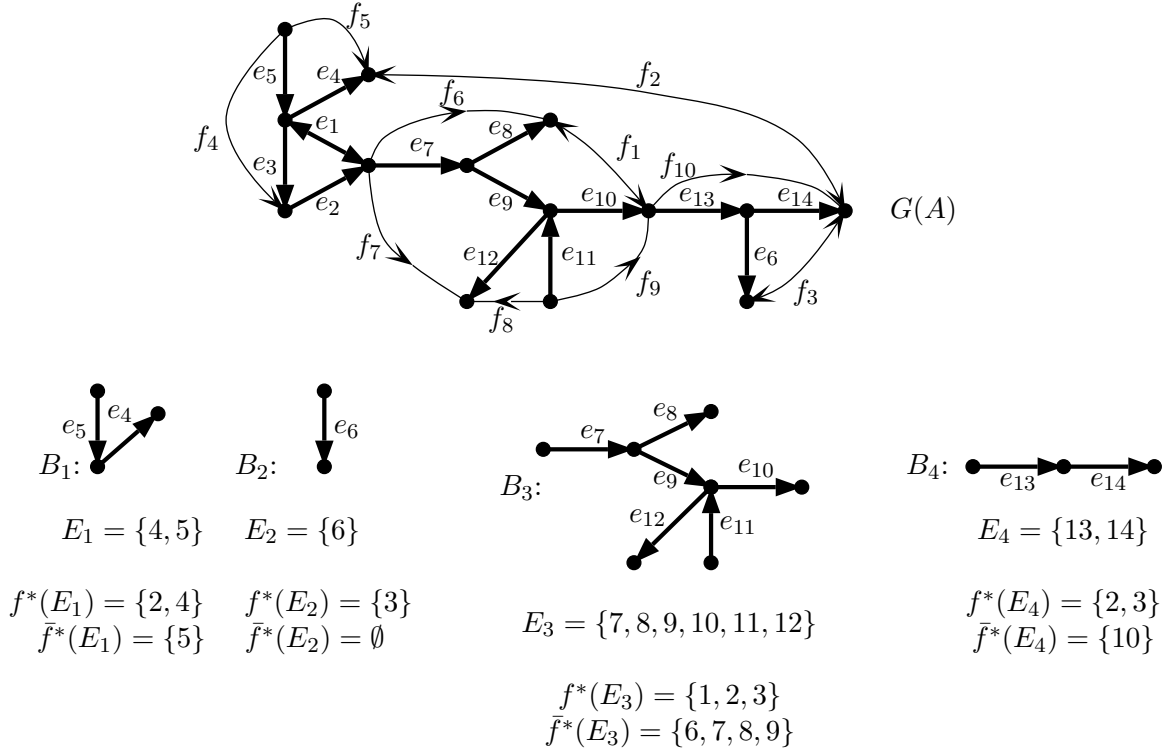
Let A be a connected matrix with entries 0, 1, 2 or $\frac{1}{2}$, and R^* a row index subset of A . If A has an R^* -cyclic, R^* -central or R^* -network representation $G(A)$, then the basic forest in $G(A)$ with edge index set $\overline{R^*}$ has some nice properties that have to be recognized. For that purpose, we construct a digraph D with respect to R^* as well as matrices, called bonsai matrices, related with the vertices of D . These objects play a significant role in the recognition of R^* -cyclic and R^* -central matrices. They are also useful for the recognition of $\{1, \rho\}$ -corelated network matrices in Section 11.4. Notice that if A is R^* -cyclic, then for any column index j such that $s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset$ we have $s_{\frac{1}{2}}(A_{\bullet j}) = R^*$ (see Lemmas 4.1 and 3.6). Further, in Chapter 11, we deal with R^* -central matrices without any $\frac{1}{2}$ -entry. So, throughout this chapter, we assume that if $s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset$ for some column index j , then $s_{\frac{1}{2}}(A_{\bullet j}) = R^*$.

Let $S^* = \{j : s(A_{\bullet j}) \cap R^* \neq \emptyset\}$. We partition $\overline{R^*}$ into subsets E_1, \dots, E_b corresponding to the vertices of D as follows. For $i, i' \in \overline{R^*}$, i and i' are in a same subset E_ℓ ($1 \leq \ell \leq b$) if and only if there exist column indexes j and j' such that $i \in s(A_{\bullet j})$, $i' \in s(A_{\bullet j'})$, and j and j' are in the same connected component of the graph $H(A_{\bullet \overline{S^*}})$. See Figure 7.1.

For all $1 \leq \ell \leq b$, we define the following objects. The set E_ℓ is called a *bonsai*. Recall that $f(E_\ell) = \{j : s(A_{\bullet j}) \cap E_\ell \neq \emptyset\}$. The set $f^*(E_\ell) = f(E_\ell) \cap S^*$ is called the *global connector set of E_ℓ* and $\bar{f}^*(E_\ell) = f(E_\ell) \cap \overline{S^*}$ the *local connector set of E_ℓ* . From the way of partitioning $\overline{R^*}$ into E_1, \dots, E_b , it results that $\bar{f}^*(E_\ell) = \{j : s(A_{\bullet j}) \subseteq E_\ell\}$. See Figure 7.2.

Suppose that A has an R^* -cyclic, R^* -central or R^* -network representation $G(A)$. Let $1 \leq \ell \leq b$. From the way of partitioning $\overline{R^*}$, it follows that the bonsai E_ℓ is the edge index set of a tree in $G(A)$, denoted as B_ℓ , which is called a *bonsai*. For any j in the local connector set $\bar{f}^*(E_\ell)$ of E_ℓ , the set $s(A_{\bullet j}) \cap E_\ell$ corresponds to the edge index set of a (basic) directed path in B_ℓ . Any column with index in the global connector set $f^*(E_\ell)$ of E_ℓ corresponds to a nonbasic edge whose fundamental circuit intersects the edge set of B_ℓ and the edge set of the basic cycle. By Lemma 4.2, for any j in the global connector set $f^*(E_\ell)$ of E_ℓ , the intersection of the fundamental circuit of f_j with B_ℓ forms either a directed path, called a *B_ℓ -path generated by f_j* , or a union of two distinct maximal directed paths called *B_ℓ -paths* and leaving a common node (see also Figures 7.8 and 7.9).

To illustrate these notions, let us study an example. In Figure 7.2, we have a $\{1, 2, 3\}$ -cyclic representation of the matrix given in Figure 7.1 and the corresponding bonsais. Observe that

Figure 7.1: A binet matrix A and the graph $H(A_{\bullet, \overline{S^*}})$.Figure 7.2: A binet representation $G(A)$ of the matrix A given in Figure 7.1 and the corresponding bonsais.

the path made up of e_{13} and e_{14} is a B_4 -path generated by f_2 and f_3 , and the path consisting of e_{13} is also a B_4 -path generated by f_3 .

Furthermore, the nonbasic edge f_1 generates two B_3 -paths, one with edge index set $\{7, 8\}$ and another one with edge index set $\{7, 9, 10\}$. This raises the following questions. Does there exist a binet representation of A such that f_1 generates a B_3 -path with edge index set not equal to $\{7, 8\}$ nor $\{7, 9, 10\}$? Actually, there is no such representation. Is it possible to compute the row index sets $\{7, 8\}$ and $\{7, 9, 10\}$ corresponding to B_3 -paths without knowing any binet representation of A ? Section 7.1 states a useful theorem in this direction. In Section 7.2, for all $1 \leq \ell \leq b$, we compute row index subsets of E_ℓ , called E_ℓ -paths, denoted as $E_\ell^1, \dots, E_\ell^{m(\ell)}$, and define a bonsai matrix N_ℓ . The matrix $A_{E_\ell \times \bar{f}^*(E_\ell)}$ and the vectors $\chi_{E_\ell^k}^{\{1, \dots, n\}}$ are submatrices of N_ℓ . We present the construction of a digraph D in Section 7.3. The vertex set of D is $\{E_1, \dots, E_b\}$. Provided that A has an R^* -cyclic representation $G(A)$, an E_ℓ -path corresponds to the edge index set of a B_ℓ -path, and an edge $(E_\ell, E_{\ell'})$ in D labeled $E_{\ell'}^k$, for some $1 \leq k \leq m(\ell')$, means the following:

- For any nonbasic edge f_β generating at least one B_ℓ -path (or equivalently, $\beta \in f^*(E_\ell)$), f_β generates a $B_{\ell'}$ -path with edge index set $E_{\ell'}^k$.
- If f_β generates two distinct B_ℓ -paths or $s_2(A_{\bullet\beta}) \neq \emptyset$, then f_β generates exactly one $B_{\ell'}$ -path with edge index set $E_{\ell'}^k = s(A_{\bullet\beta}) \cap E_{\ell'} = s_2(A_{\bullet\beta}) \cap E_{\ell'}$.

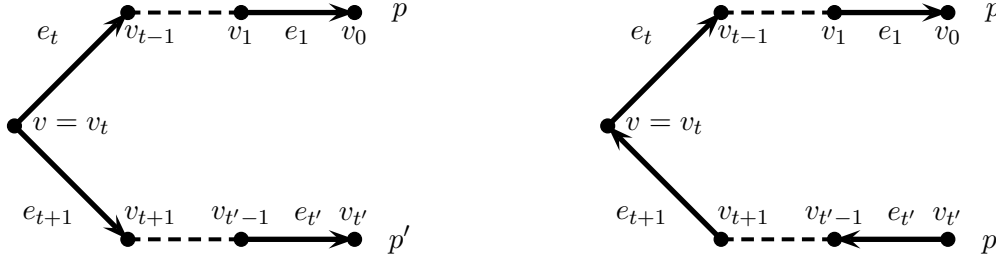
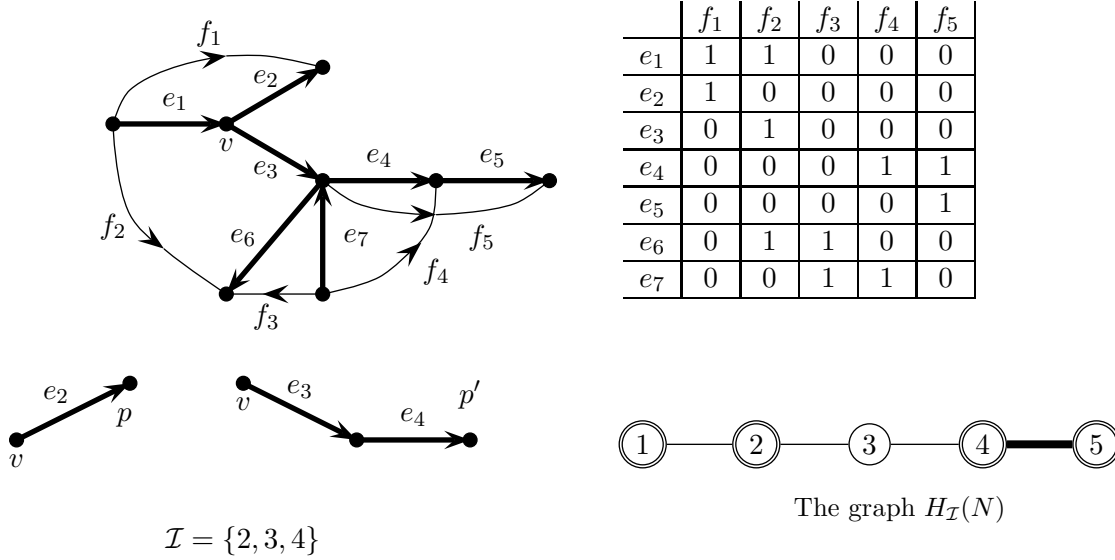
In Figure 7.2, consider B_3 and B_4 . We have $f^*(E_4) = \{2, 3\}$. The nonbasic edge f_2 generates exactly one B_4 -path with edge index set $\{13, 14\}$ and a B_3 -path with edge index set $\{7, 9, 10\}$. Moreover, f_3 generates two B_4 -paths with edge index sets $\{13, 14\}$ and $\{13\}$, respectively, and a B_3 -path with edge index set $\{7, 9, 10\} = s(A_{\bullet 3}) \cap E_3 = s_2(A_{\bullet 3}) \cap E_3$. Thus (E_4, E_3) is an edge in D labeled by $\{7, 9, 10\}$.

On the other hand, observe that B_4 and B_2 have a common node, corresponding to the endnode of a B_4 -path with edge index set $\{13\}$; and B_4 is closer to the basic cycle than B_2 . This implies that (E_2, E_4) is an edge in D labeled by $\{13\}$. More generally, $G(A)$ induces a spanning forest of D denoted by $T_{G(A)}$ and defined as follows: for any $B_\ell, B_{\ell'} \subseteq G(A)$, $(E_\ell, E_{\ell'})_{E_{\ell'}^k} \in T_{G(A)}$ if and only if one node of B_ℓ coincides with the endnode of the $B_{\ell'}$ -path with edge index set $E_{\ell'}^k$ (and $B_{\ell'}$ is closer than B_ℓ to the basic cycle). This forest is studied in Section 7.4 and motivates the definition of a feasible spanning forest of D . Section 7.5 deals with a procedure computing (in some cases) a feasible spanning forest of D .

7.1 An important theorem

Let N be a connected nonnegative network matrix, $G(N)$ a network representation of N and \mathcal{I} a row index subset of N . We suppose that \mathcal{I} is the edge index set of two (basic) disjoint directed paths p and p' having exactly one common node v in $G(N)$ which is an endnode of both p and p' as in Figure 7.3. In this section, we provide a theorem which allows us to distinguish the edge indexes of p from those of p' , by using the matrix N only.

Up to row permutations and reversing the orientation of all edges if necessary, we may suppose that the vertices of p are $v_0, v_1, \dots, v_t = v$, those of p' are $v_t = v, v_{t+1}, \dots, v_{t'}$, and $e_i = (v_i, v_{i-1})$ for $i = 1, \dots, t$; moreover, either $e_i = (v_{i-1}, v_i)$ for all $i = t+1, \dots, t'$ or

Figure 7.3: the paths p and p' in two possible configurations.Figure 7.4: A network matrix with a network representation, two directed paths p and p' starting at v and the graph $H_{\mathcal{I}}(N)$. (Heavy edges correspond to blue edges and vertices with a double circle to blue ones.)

$e_i = (v_i, v_{i-1})$ for all $i = t+1, \dots, t'$. See Figure 7.3. A path h in $H(N)$ between two nodes k and k' is said to be *minimal* if any two non-consecutive vertices of h are not adjacent in $H(N)$. We denote by $H_{\mathcal{I}}(N)$ the graph $H(N)$ in which each vertex k is blue if $s(N_{\bullet k}) \cap \mathcal{I} \neq \emptyset$ and yellow otherwise, and each edge (k, k') has a blue color if $s(N_{\bullet k}) \cap s(N_{\bullet k'}) \cap \mathcal{I} \neq \emptyset$ and yellow otherwise. See Figure 7.4. Using these preliminaries, we can state the main theorem.

Theorem 7.1 *Let k and k' be two blue vertices in $H_{\mathcal{I}}(N)$. Then $\mathcal{I} \cap s(N_{\bullet k})$ and $\mathcal{I} \cap s(N_{\bullet k'})$ are edge index subsets of a same basic directed path of $G(N)$ if and only if all minimal paths between k and k' in $H_{\mathcal{I}}(N)$ have an even number of yellow edges.*

A *blue* path is a path all of whose vertices and edges are blue. A *yellow* path is a path whose endnodes are blue and all inner vertices and edges yellow. For proving Theorem 7.1, we view any path in $H_{\mathcal{I}}(N)$ as a succession of blue and yellow paths. So we need the following lemma.

Lemma 7.2 *Let (k, k') be a blue edge in $H_{\mathcal{I}}(N)$. Then $\mathcal{I} \cap s(N_{\bullet k})$ and $\mathcal{I} \cap s(N_{\bullet k'})$ are edge index subsets of a same basic directed path of $G(N)$.*

Proof. Since N is nonnegative, $\mathcal{I} \cap s(N_{\bullet k})$ and $\mathcal{I} \cap s(N_{\bullet k'})$ are edge index sets of (basic) directed paths included in $p \cup p'$. If k and k' are linked by a blue edge, then the fundamental cycles of f_k and $f_{k'}$ have a common edge contained in p or p' , hence $\mathcal{I} \cap s(N_{\bullet k})$ and $\mathcal{I} \cap s(N_{\bullet k'})$ are edge index subsets of a same directed path. ■

Let j and j' be two blue vertices in $H_{\mathcal{I}}(N)$ and h' a yellow minimal path from j to j' . Up to column permutations, we may note $h' = (j = 1, (1, 2), 2, \dots, j')$. Since h' is a yellow minimal path, it follows that $s(N_{\bullet 1}) \cap s(N_{\bullet j'}) \cap \mathcal{I} = \emptyset$. Then, using the ordering of the edges in $p \cup p'$, we may suppose that $i_{\max} = \max\{i : i \in s(N_{\bullet 1}) \cap \mathcal{I}\} < i_{\min} = \min\{i : i \in s(N_{\bullet j'}) \cap \mathcal{I}\}$ (the case $\min\{i : i \in s(N_{\bullet 1}) \cap \mathcal{I}\} > \max\{i : i \in s(N_{\bullet j'}) \cap \mathcal{I}\}$ is similar for the proof of next lemmas). We denote by T the basic subgraph of $G(N)$ with edge index set $\cup_{\beta=1}^{j'} s(N_{\bullet \beta})$. Clearly, T is connected. The following lemma asserts that $e_{i_{\max}}$ and $e_{i_{\min}}$ are adjacent.

Lemma 7.3 *The edges $e_{i_{\max}}$ and $e_{i_{\min}}$ are adjacent in T .*

Proof. Suppose, to the contrary, that $e_{i_{\max}}$ and $e_{i_{\min}}$ are not adjacent. Then there would exist a basic edge, say e_{i_0} , with $i_0 \in R$ between $e_{i_{\max}}$ and $e_{i_{\min}}$ in T . By definition of $e_{i_{\max}}$ and $e_{i_{\min}}$, e_{i_0} would not be in the fundamental cycle of f_1 and $f_{j'}$. Furthermore, since β is yellow for all $1 < \beta < j'$, e_{i_0} would not be in the fundamental cycle of f_β for all $1 < \beta < j'$. Therefore $e_{i_{\max}}, e_{i_{\min}} \in T$ and $e_{i_0} \notin T$. Thus T would not be connected, which is a contradiction. ■

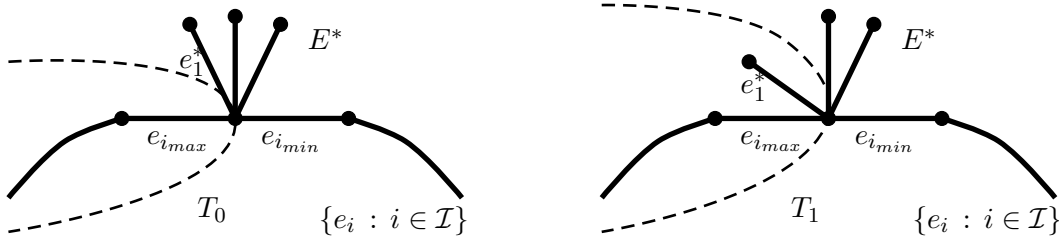


Figure 7.5: An illustration of T_0 and T_1 .

Let E^* be the set of basic edges in $G(N)$ incident to $v_{i_{\max}}$. From the following lemma we may deduce that the fundamental cycle of any edge with index in h' contains $v_{i_{\max}}$.

Lemma 7.4 *For $1 \leq \beta \leq j'$, the set $s(N_{\bullet \beta})$ contains the indexes of one edge entering $v_{i_{\max}}$ and one edge leaving $v_{i_{\max}}$. Moreover, for $1 \leq \beta < j'$, there is exactly one edge of E^* with index in $s(N_{\bullet \beta}) \cap s(N_{\bullet \beta+1})$.*

Proof. We prove the lemma by induction on β (with $1 \leq \beta \leq j'$). Let T_0 be the connected component of $(T - E^* + \{e_{i_{\max}}\})$ containing $e_{i_{\max}}$. See Figure 7.5. Suppose by contradiction that the basic edges with index in $s(N_{\bullet 1})$ are all in T_0 . Let $\delta_0 = \min\{\beta \in \mathbb{N} : \{e_i : i \in s(N_{\bullet \beta+1})\} \not\subseteq T_0\}$. Clearly, $1 \leq \delta_0 < j'$. As $s(N_{\bullet \delta_0}) \cap s(N_{\bullet \delta_0+1}) \neq \emptyset$,

we deduce that $i_{max} \in s(N_{\bullet\delta_0+1})$. So $(1, \delta_0 + 1)$ is a blue edge in $H_{\mathcal{I}}(N)$, which contradicts the fact that h' is either minimal or without any blue edge. Let $e_1^* (\neq e_{i_{max}})$ be the edge belonging to E^* and the fundamental cycle of f_1 . Denote by T_1 the connected component of $(T - E^* + \{e_{i_{max}}, e_1^*\})$ containing e_1^* (and $e_{i_{max}}$). Notice that $e_1^* \neq e_{i_{min}}$ since $(1, j')$ is not blue. Hence, if $\{e_i : i \in s(N_{\bullet 2})\} \subseteq T_1$, then the node 1 would be adjacent to $\min\{\beta \in h' : \{e_i : i \in s(N_{\bullet\beta})\} \not\subseteq T_1\} \neq 2$ in $H_{\mathcal{I}}(N)$, contradicting the minimality of h' . Since $(1, 2)$ is yellow, it results that the fundamental cycle of f_2 contains e_1^* and an edge in $E^* - T_1$.

Let $1 \leq \beta_0 < j'$ and suppose that the lemma is true for $\beta = 1, \dots, \beta_0$. Denote by $e_{\beta_0}^*$ and $e_{\beta_0'}^*$ the edges of E^* in the fundamental cycle of f_{β_0} . Let T_{β_0} be the connected component of $(T - E^* + \{e_{\beta_0}^*, e_{\beta_0'}^*\})$ containing $e_{\beta_0}^*$. As previously, if $\{e_i : i \in s(N_{\beta_0+1})\} \subseteq T_{\beta_0}$, then β_0 would be adjacent to a vertex of h' distinct from $\beta_0 + 1$, a contradiction. So, exactly one of the edges $e_{\beta_0}^*$ and $e_{\beta_0'}^*$ with one other edge of E^* are in the fundamental cycle of f_{β_0+1} . Since N is nonnegative, for $1 \leq \beta \leq j'$, one both edges belonging to the fundamental cycle of f_{β} and E^* is entering $v_{i_{max}}$ and the other is leaving $v_{i_{max}}$. This completes the proof. ■

The proof of Theorem 7.1 is based on the following result (see Figure 7.6).

Lemma 7.5 *The sets $\mathcal{I} \cap s(N_{\bullet j})$ and $\mathcal{I} \cap s(N_{\bullet j'})$ are edge index subsets of the same basic directed path of $G(N)$ if and only if h' has an even number of (yellow) edges. If h' has an odd number of yellow edges, then the index t (resp., $t + 1$) belongs to $\mathcal{I} \cap s(N_{\bullet j})$ (resp., $\mathcal{I} \cap s(N_{\bullet j'})$) or vice versa, and the paths p and p' are both leaving v .*

Proof. Assume that $e_{i_{max}}$ is leaving $v_{i_{max}}$. By Lemma 7.4, it results that $e_{i_{min}}$ is entering $v_{i_{max}}$ if and only if h' has an even length. We deduce that h' has an even length if and only if $\mathcal{I} \cap s(N_{\bullet 1})$ and $\mathcal{I} \cap s(N_{\bullet j'})$ are edge index subsets of a same directed path. In the case where h' has an odd length, we have $e_{i_{max}} = e_t$ and $e_{i_{min}} = e_{t+1}$ (since $i_{max} < i_{min}$). See Figure 7.6. ■

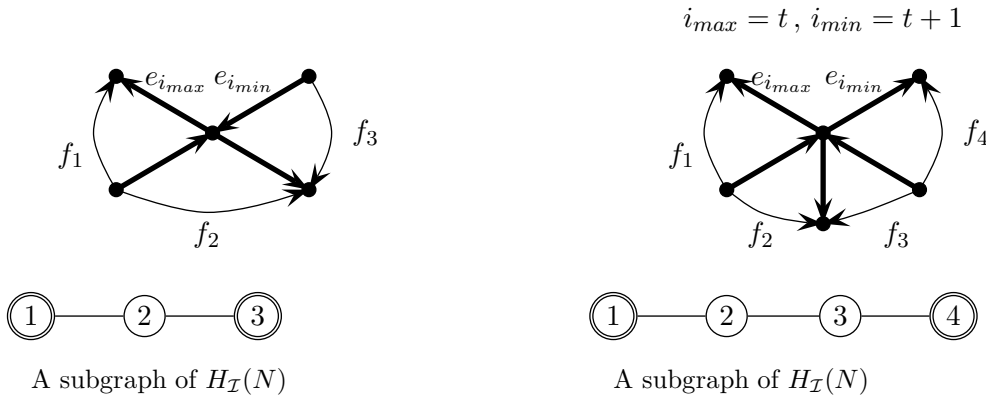


Figure 7.6: An illustration of Lemma 7.5 in the case where h' has an even length (on the left) and an odd length (on the right).

We are now able to prove Theorem 7.1.

Proof of Theorem 7.1. Let h be a minimal path between k and k' . By minimality, h has at most one yellow subpath of odd length. Otherwise, using Lemma 7.5, t or $t+1$ would be in the support of two non-consecutive vertices of h , a contradiction. Viewing h as a succession of blue subpaths and yellow subpaths, it follows from Lemmas 7.2 and 7.5 that h has an even number of yellow edges if and only if $\mathcal{I} \cap s(N_{\bullet k})$ and $\mathcal{I} \cap s(N_{\bullet k'})$ are edge index subsets of a same (basic) directed path. \blacksquare

In Figure 7.4, $\mathcal{I} \cap s(N_{\bullet 1})$ and $\mathcal{I} \cap s(N_{\bullet 4})$ are not edge index subsets of a same directed path, because the minimal path from $\textcircled{1}$ to $\textcircled{4}$ in $H_{\mathcal{I}}(N)$ has three yellow edges. But $\mathcal{I} \cap s(N_{\bullet 2})$ and $\mathcal{I} \cap s(N_{\bullet 5})$ are edge index subsets of a same directed path (p'), since the minimal path from $\textcircled{2}$ to $\textcircled{5}$ in $H_{\mathcal{I}}(N)$ has two yellow edges.

7.2 Bonsai matrices

Let A be a nonnegative connected matrix, R^* a row index subset of A and E_1, \dots, E_b a partitioning of $\overline{R^*}$ as described in the introduction of the chapter. The object of the present section is to compute subsets of E_ℓ called E_ℓ -paths, and to define a bonsai matrix associated with E_ℓ , for all $1 \leq \ell \leq b$. We will also prove some results about bonsai matrices.

For all $1 \leq \ell \leq b$ and $\beta \in f^*(E_\ell)$, let us compute subsets of E_ℓ as follows. If $|E_\ell| > 1$, then by the way of partitioning $\overline{R^*}$, for all $i \in E_\ell$, there exists a column index $j_0 \in \bar{f}^*(E_\ell)$ such that $i \in s(A_{\bullet j_0})$. Let us consider the following procedure.

Procedure $E\ell\text{path}(A, E_\ell, \beta)$

Input: A matrix A , a bonsai E_ℓ ($1 \leq \ell \leq b$) and some $\beta \in f^*(E_\ell)$.

Output: Row index subsets $E_\ell^I(A_{\bullet \beta})$ and $E_\ell^{II}(A_{\bullet \beta})$ of E_ℓ .

let $\mathcal{I} = s_1(A_{\bullet \beta}) \cap E_\ell$;

if $|\mathcal{I}| > 1$, **then**

let $j_0 \in \bar{f}^*(E_\ell)$ such that $s(A_{\bullet j_0}) \cap \mathcal{I} \neq \emptyset$ and $S_0 = \{j_0\}$;

compute a spanning tree $T_{\mathcal{I}}$ in $H_{\mathcal{I}}(A_{E_\ell \times \bar{f}^*(E_\ell)})$ rooted at j_0 such that

for each $j \in V(T_{\mathcal{I}})$, the path from j to j_0 in $T_{\mathcal{I}}$ is minimal in $H_{\mathcal{I}}(A_{E_\ell \times \bar{f}^*(E_\ell)})$;

for every $j \in \bar{f}^*(E_\ell)$ such that $s(A_{\bullet j}) \cap \mathcal{I} \neq \emptyset$, **do**

if the path from j_0 to j in $T_{\mathcal{I}}$ has an even number of yellow edges,

then add j in S_0 ;

endfor

let $\mathcal{I}^I = s(A_{\bullet S_0}) \cap \mathcal{I}$ and $\mathcal{I}^{II} = \mathcal{I} \setminus \mathcal{I}^I$;

otherwise

let $\mathcal{I}^I = \mathcal{I}$ and $\mathcal{I}^{II} = \emptyset$;

endif

let $E_\ell^I(A_{\bullet \beta}) = \{i \in E_\ell : A_{i\beta} = 2\} \cup \mathcal{I}^I$, $E_\ell^{II}(A_{\bullet \beta}) = \{i \in E_\ell : A_{i\beta} = 2\} \cup \mathcal{I}^{II}$;

output $E_\ell^I(A_{\bullet \beta})$ and $E_\ell^{II}(A_{\bullet \beta})$;

For all $1 \leq \ell \leq b$ and $\beta \in f^*(E_\ell)$, since $s_{\frac{1}{2}}(A_{\bullet \beta}) = R^*$ by assumption (see page 83), we observe that $s_1(A_{\bullet \beta}) \neq \emptyset$ or $s_2(A_{\bullet \beta}) \neq \emptyset$, hence $E_\ell^I(A_{\bullet \beta}) \neq \emptyset$. The procedure above is motivated by the following lemma.

Lemma 7.6 *Suppose that A has an R^* -cyclic, R^* -central or R^* -network representation $G(A)$. Then for any bonsai E_ℓ ($1 \leq \ell \leq b$) and $\beta \in f^*(E_\ell)$, the row index subsets $E_\ell^I(A_{\bullet\beta})$ and $E_\ell^{II}(A_{\bullet\beta})$ (if not empty) output by the procedure Elpath are the edge index sets of B_ℓ -paths generated by f_β .*

Proof. Suppose that A is R^* -cyclic (the cases R^* -central or R^* -network are similar or simpler). Let $1 \leq \ell \leq b$, $\beta \in f^*(E_\ell)$ and $\mathcal{I} = s_1(A_{\bullet\beta}) \cap E_\ell$. Notice that if $|\mathcal{I}| > 1$, then the way of partitioning $\overline{R^*}$ implies that for all $i \in \mathcal{I}$ there exists some $j_0 \in \bar{f}^*(E_\ell)$ such that $A_{ij_0} = 1$. By Lemmas 4.1 and 4.2 and the description of the different types of fundamental circuits given in page 41 (see Figure 4.4), it results that \mathcal{I} is the edge index set of a directed path, say p , or two directed paths, say p and p' , having their initial node in common. Furthermore, if $s_2(A_{\bullet\beta}) \cap E_\ell \neq \emptyset$, then the set $s_2(A_{\bullet\beta}) \cap E_\ell$ is the edge index set of a directed path, say p'' , and the initial node of p and p' coincides with the terminal node of p'' (the terminal node of p'' is the unique node of p'' in common with p and p'). Then the proof of Lemma 7.6 follows from Theorem 7.1. \blacksquare

For any bonsai E_ℓ , let $E_\ell^I(A_{\bullet\beta})$ and $E_\ell^{II}(A_{\bullet\beta})$ be output by the procedure Elpath and $m(\ell)$ the cardinality of the set $\{E_\ell^I(A_{\bullet\beta}), E_\ell^{II}(A_{\bullet\beta}) : \beta \in f^*(E_\ell)\} \setminus \{\emptyset\}$, and denote by $E_\ell^1, E_\ell^2, \dots, E_\ell^{m(\ell)}$ all distinct elements of this set. For all $1 \leq k \leq m(\ell)$, E_ℓ^k is called an E_ℓ -path, and let $f^*(E_\ell^k) = \{\beta \in f^*(E_\ell) : E_\ell^k = E_\ell^I(A_{\bullet\beta}) \text{ or } E_\ell^{II}(A_{\bullet\beta})\}$. By Lemma 7.6, if A has an R^* -cyclic, R^* -central or R^* -network representation $G(A)$, then for any $1 \leq \ell \leq b$, $1 \leq k \leq m(\ell)$ and $\beta \in f^*(E_\ell^k)$, the E_ℓ -path E_ℓ^k is the edge index set of a B_ℓ -path generated by f_β , denoted as B_ℓ^k . For any $\beta \in f^*(E_\ell^k)$, we say that $A_{\bullet\beta}$ generates the E_ℓ -path E_ℓ^k . See Figure 7.7.

Given the set $\{E_\ell^1, \dots, E_\ell^{m(\ell)}\}$ of E_ℓ -paths for some $1 \leq \ell \leq b$, the following procedure partitions this set into two classes. A graphical interpretation of these classes is given in Lemma 7.7 below.

Procedure TwoClasses($A, \{E_\ell^1, \dots, E_\ell^{m(\ell)}\}$)

Input: A matrix A and a set of E_ℓ -paths $\{E_\ell^1, \dots, E_\ell^{m(\ell)}\}$ for some $1 \leq \ell \leq b$.

Output: Two sets J_ℓ^1 and J_ℓ^2 such that $\{E_\ell^1, \dots, E_\ell^{m(\ell)}\} = J_\ell^1 \uplus J_\ell^2$.

let $J_\ell^1 = \{E_\ell^1\}$; $J_\ell^2 = \{\}$ and $j_1 \in \bar{f}^*(E_\ell)$ such that $s(A_{\bullet j_1}) \cap E_\ell^1 \neq \emptyset$;

compute a spanning tree T in $H(A_{E_\ell \times \bar{f}^*(E_\ell)})$ rooted at j_1 such that for each $j \in V(T)$

the path from j to j_1 in T is minimal in $H(A_{E_\ell \times \bar{f}^*(E_\ell)})$;

for $k = 2, \dots, m(\ell)$, **do**

let $j_k \in V(T)$ such that $s(A_{\bullet j_k}) \cap E_\ell^k \neq \emptyset$ and h be the path from j_1 to j_k in T ;

color h as if $T \subseteq H_{E_\ell^1 \cup E_\ell^k}(A_{E_\ell \times \bar{f}^*(E_\ell)})$;

if $E_\ell^1 \cap E_\ell^k \neq \emptyset$ or h has an odd number of yellow edges, **then**

let $J_\ell^1 = J_\ell^1 \cup \{E_\ell^k\}$;

otherwise

let $J_\ell^2 = J_\ell^2 \cup \{E_\ell^k\}$;

endif

endfor

output J_ℓ^1 and J_ℓ^2 ;

For all $1 \leq \ell \leq b$, let J_ℓ^1 and J_ℓ^2 be output by the procedure **TwoClasses**; for any E_ℓ -paths

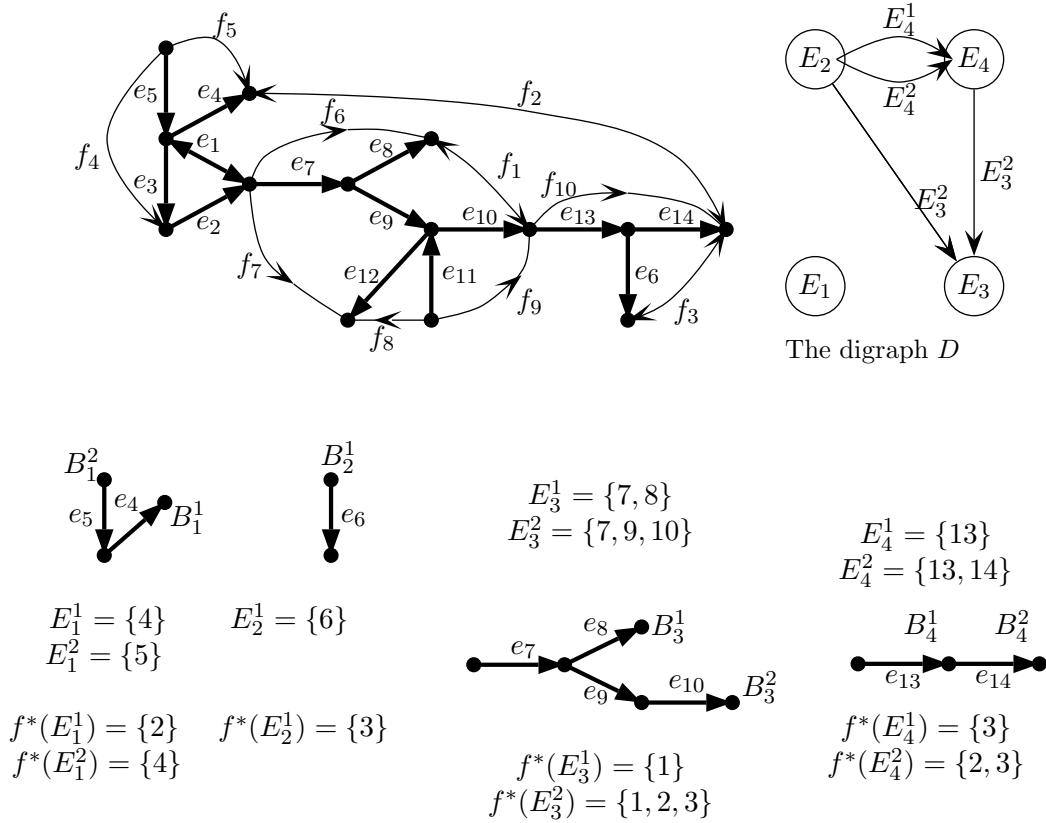


Figure 7.7: An R^* -cyclic representation of the matrix A given in Figure 7.1, the digraph D and all computed E_ℓ -paths and corresponding B_ℓ -paths, where E_1, \dots, E_4 are defined in Figure 7.1.

E_ℓ^k and $E_\ell^{k'}$ ($1 \leq k, k' \leq m(\ell)$), we define the following relation: $E_\ell^k \sim_{E_\ell} E_\ell^{k'}$ if and only if E_ℓ^k and $E_\ell^{k'}$ belong to the same class (J_ℓ^1 or J_ℓ^2). In Figure 7.7, E_1^1 and E_1^2 are two E_1 -paths. Since $s(A_{\bullet 5}) \cap E_1^k \neq \emptyset$ for $k = 1$ and 2 and the constant path at node 5 in $H_{E_1^1 \cup E_1^2}(A_{E_1 \times \bar{f}^*(E_1)})$ has zero yellow edges, $E_1^1 \sim_{E_1} E_1^2$ and $J_1^k = \{E_1^k\}$ for $k = 1$ and 2 .

Lemma 7.7 Suppose that A has an R^* -cyclic, R^* -central or R^* -network representation $G(A)$. Let $1 \leq \ell \leq b$ and suppose that all B_ℓ -paths in $G(A)$ have a common endnode say v_ℓ . Then for all $1 \leq k, k' \leq m(\ell)$, the B_ℓ -paths B_ℓ^k and $B_\ell^{k'}$ both enter, or leave v_ℓ if and only if $E_\ell^k \sim_{E_\ell} E_\ell^{k'}$.

Proof. By assumption, all B_ℓ -paths in $G(A)$ have a common endnode v_ℓ . Let $1 \leq k, k' \leq m(\ell)$. By Theorem 7.1, the B_ℓ -paths B_ℓ^k and $B_\ell^{k'}$ both enter, or leave v_ℓ if and only if $E_\ell^k \cap E_\ell^{k'} \neq \emptyset$, or for any j, j' such that $s(A_{\bullet j}) \cap E_\ell^k \neq \emptyset$ and $s(A_{\bullet j'}) \cap E_\ell^{k'} \neq \emptyset$, all minimal paths in $H_{E_\ell^k \cup E_\ell^{k'}}(A_{E_\ell \times \bar{f}^*(E_\ell)})$ between j and j' have an odd number of yellow edges. Hence, by construction of J_ℓ^1 and J_ℓ^2 , the B_ℓ -paths B_ℓ^k and $B_\ell^{k'}$ both enter, or leave v_ℓ if and only if $E_\ell^k \sim_{E_\ell} E_\ell^{k'}$. \blacksquare

For all $1 \leq \ell \leq b$, let $n(\ell)$ denote the cardinality of the set J_ℓ^1 . Without loss of generality, we may assume that $J_\ell^1 = \{E_\ell^1, \dots, E_\ell^{n(\ell)}\}$. Notice that in general the bipartition $J_\ell^1 \uplus J_\ell^2$ given by the procedure TwoClasses may be not unique.

Now we can define bonsai matrices. For all $1 \leq \ell \leq b$, if $J_\ell^2 = \emptyset$, then let

$$N_\ell = \begin{pmatrix} A_{E_\ell \times \bar{f}^*(E_\ell)} & \chi_{E_\ell^1}^{E_\ell} \cdots \chi_{E_\ell^{n(\ell)}}^{E_\ell} \\ \mathbf{0}_{1 \times |\bar{f}^*(E_\ell)|} & \mathbf{1}_{1 \times n(\ell)} \end{pmatrix};$$

otherwise ($J_\ell^1 \neq \emptyset$ and $J_\ell^2 \neq \emptyset$), let

$$N_\ell = \begin{pmatrix} A_{E_\ell \times \bar{f}^*(E_\ell)} & \chi_{E_\ell^1}^{E_\ell} \cdots \chi_{E_\ell^{n(\ell)}}^{E_\ell} & \chi_{E_\ell^{n(\ell)+1}}^{E_\ell} \cdots \chi_{E_\ell^{m(\ell)}}^{E_\ell} & \mathbf{0}_{|E_\ell| \times 1} \\ \mathbf{0}_{1 \times |\bar{f}^*(E_\ell)|} & \mathbf{1}_{1 \times n(\ell)} & \mathbf{0}_{1 \times (m(\ell) - n(\ell))} & 1 \\ \mathbf{0}_{1 \times |\bar{f}^*(E_\ell)|} & \mathbf{0}_{1 \times n(\ell)} & \mathbf{1}_{1 \times (m(\ell) - n(\ell))} & 1 \end{pmatrix}.$$

The matrix N_ℓ is called the *bonsai* matrix associated with the bonsai E_ℓ . Any row of N_ℓ of type $[A_{\{i\} \times \bar{f}^*(E_\ell)} \cdots]$ for some $i \in E_\ell$ is indexed by i . A row (respectively, a column) of N_ℓ that is not indexed by an element in E_ℓ (respectively, $\bar{f}^*(E_\ell)$) is said to be *artificial* as well as the corresponding edge in any network representation of N_ℓ (provided that such a representation exists). In the case where $R^* = \{i^*\}$ and $J_\ell^2 = \emptyset$, the index of the artificial row of N_ℓ is denoted as i^* .

Lemma 7.8 *Let $1 \leq \ell \leq b$ and suppose that A has an R^* -cyclic, R^* -central or R^* -network representation $G(A)$. If all B_ℓ -paths in $G(A)$ have a common endnode v_ℓ , then the bonsai matrix N_ℓ is a network matrix.*

Proof. Suppose first that $J_\ell^2 \neq \emptyset$. Let us denote B'_ℓ the subtree B_ℓ of $G(A)$ with two more basic directed edges (one entering v_ℓ and the other one leaving v_ℓ). Since v_ℓ is an endnode of every B_ℓ -path, by Lemma 7.7, we deduce that each column of N_ℓ is the edge incidence vector of a directed path in B'_ℓ .

Now suppose that $J_\ell^2 = \emptyset$. This implies that either all B_ℓ -paths of $G(A)$ enter v_ℓ or leave v_ℓ . Let B'_ℓ be the subtree B_ℓ of $G(A)$ with one more basic directed edge which either enters v_ℓ (if all B_ℓ -paths leave v_ℓ) or leaves v_ℓ (otherwise) and we conclude as before. \blacksquare

Let $1 \leq \ell \leq b$ and suppose that N_ℓ has a basic network representation $G(N_\ell)$. In the case where $J_\ell^2 = \emptyset$, since $A_{E_\ell \times \bar{f}^*(E_\ell)}$ is a connected matrix, the artificial edge, call it \tilde{e} , is an end-edge in $G(N_\ell)$. We denote by v_ℓ the endnode of \tilde{e} which is a cutvertex. In the case where $J_\ell^2 \neq \emptyset$, due to the last column of N_ℓ , we may denote by v_ℓ the node in $G(N_\ell)$ incident with both artificial edges.

Lemma 7.9 *Let $1 \leq \ell \leq b$ and suppose that the bonsai matrix N_ℓ has a network representation $G(N_\ell)$. Then, for all $1 \leq k \leq m(\ell)$, E_ℓ^k is the edge index set of a path with one endnode equal to v_ℓ . Moreover, for any $1 \leq k, k' \leq m(\ell)$, both paths in $G(N_\ell)$ with edge index sets E_ℓ^k and $E_\ell^{k'}$ enter or leave v_ℓ if and only if $E_\ell^k \sim_{E_\ell} E_\ell^{k'}$.*

Proof. Suppose $J_\ell^2 \neq \emptyset$ (the case $J_\ell^2 = \emptyset$ is similar and simpler). Up to a reversing of the orientation of all edges, we may suppose that the first artificial edge goes from a node x to

v_ℓ and the second one (corresponding to the last row) is equal to (v_ℓ, y) . If x is a node of B_ℓ , then each column of the matrix $\begin{pmatrix} \chi_{E_\ell^{n(\ell)+1}}^{E_\ell} \cdots \chi_{E_\ell^{m(\ell)}}^{E_\ell} \\ \mathbf{0}_{1 \times (m(\ell)-n(\ell))} \\ \mathbf{1}_{1 \times (m(\ell)-n(\ell))} \end{pmatrix}$ can not be the edge incidence vector of a path in $G(N_\ell)$. We have a similar contradiction with $y \in B_\ell$ instead of $x \in B_\ell$. So v_ℓ belongs to B_ℓ . Since (x, v_ℓ) is entering v_ℓ and $[(\chi_{E_\ell^k}^{E_\ell})^T \mathbf{1} \mathbf{0}]^T$ is the incidence vector of a path for $k = 1, \dots, n(\ell)$, it follows that E_ℓ^k is the edge index set of a path leaving v_ℓ for $k = 1, \dots, n(\ell)$. Similarly, we prove that for $k = n(\ell) + 1, \dots, m(\ell)$, E_ℓ^k is the edge index set of a path entering v_ℓ . ■

If the matrix N_ℓ has a network representation $G(N_\ell)$, by Lemma 7.9 and for ease of notation, the path in $G(N_\ell)$ with edge index set E_ℓ^k ($1 \leq k \leq m(\ell)$) is called a B_ℓ -path. Moreover, the basic connected subgraph of $G(N_\ell)$ with edge index set E_ℓ is called a *bonsai* and is denoted B_ℓ . We say that the bonsai B_ℓ is a v_ℓ -rooted network representation of N_ℓ . Given N_ℓ and a v_ℓ -rooted network representation of N_ℓ it is easy to construct a network representation of N_ℓ .

Lemma 7.10 *Let $1 \leq \ell \leq b$ and $1 \leq k, k' \leq m(\ell)$. Suppose that the bonsai matrix N_ℓ has a v_ℓ -rooted network representation B_ℓ . Suppose also that the matrix $A_{E_\ell \times \bar{f}^*(E_\ell)}$ has a network representation $G(A_{E_\ell \times \bar{f}^*(E_\ell)})$ in which E_ℓ^k and $E_\ell^{k'}$ are edge index sets of directed paths having exactly one common endnode, say z_ℓ . Then the paths with edge index sets E_ℓ^k and $E_\ell^{k'}$ in $G(A_{E_\ell \times \bar{f}^*(E_\ell)})$ both enter, or leave z_ℓ if and only if $E_\ell^k \sim_{E_\ell} E_\ell^{k'}$.*

Proof. By Theorem 7.1, the B_ℓ -paths B_ℓ^k and $B_\ell^{k'}$ in B_ℓ both enter, or leave v_ℓ if and only if $E_\ell^k \cap E_\ell^{k'} \neq \emptyset$, or for any j, j' such that $s(A_{\bullet j}) \cap E_\ell^k \neq \emptyset$ and $s(A_{\bullet j'}) \cap E_\ell^{k'} \neq \emptyset$, all minimal paths in $H_{E_\ell^k \cup E_\ell^{k'}}(A_{E_\ell \times \bar{f}^*(E_\ell)})$ between j and j' have an odd number of yellow edges. Similarly, the basic paths in $G(A_{E_\ell \times \bar{f}^*(E_\ell)})$ with edge index set E_ℓ^k and $E_\ell^{k'}$ both enter, or leave z_ℓ if and only if $E_\ell^k \cap E_\ell^{k'} \neq \emptyset$, or for any j, j' such that $s(A_{\bullet j}) \cap E_\ell^k \neq \emptyset$ and $s(A_{\bullet j'}) \cap E_\ell^{k'} \neq \emptyset$, all minimal paths in $H_{E_\ell^k \cup E_\ell^{k'}}(A_{E_\ell \times \bar{f}^*(E_\ell)})$ between j and j' have an odd number of yellow edges. Using Lemma 7.9, this completes the proof. ■

7.3 A digraph D

Let A be a connected $\{0, 1, 2, \frac{1}{2}\}$ -matrix of size $n \times m$ and R^* a row index subset of A . Let us denote by α the number of nonzero elements in A . We assume that $\overline{R^*}$ has been partitioned into subsets E_1, \dots, E_b and for all $\ell = 1, \dots, b$, the classes J_ℓ^1 and J_ℓ^2 have been computed in Section 7.2. We give the definition of a digraph denoted by D and prove related results.

The vertex set of D is $V = \{E_1, \dots, E_b\}$ and its edge index set is denoted by Υ ($D = (V, \Upsilon)$). Up to column permutations, we may assume $S^* = \{1, \dots, s\}$ where $s = |S^*|$. Let $g: V \cup_{\ell, k} E_\ell^k \rightarrow \{0, 1, 2\}^s$ be the function given by

$$g_\beta(E_\ell^k) = \begin{cases} 2 & \text{if } E_\ell^k = E_\ell^I(A_{\bullet\beta}) = E_\ell^{II}(A_{\bullet\beta}) \\ 1 & \text{if } E_\ell^k = E_\ell^i(A_{\bullet\beta}) \text{ (} i = I \text{ or } II \text{) and } E_\ell^I(A_{\bullet\beta}) \neq E_\ell^{II}(A_{\bullet\beta}) \\ 0 & \text{otherwise} \end{cases}$$

and $g_\beta(E_\ell) = \sum_{h=1}^{m(\ell)} g_\beta(E_\ell^h)$, for all $1 \leq \ell \leq b$, $1 \leq k \leq m(\ell)$ and $\beta \in S^*$. Observe that for all $1 \leq \ell \leq b$ and $1 \leq k \leq m(\ell)$, we have the inequality $g(E_\ell) \geq g(E_\ell^k)$. For $1 \leq \ell, \ell' \leq b$ and $1 \leq k \leq m(\ell)$, there is an arc $(E_\ell, E_{\ell'}) \in \Upsilon$ labeled $E_{\ell'}^k$ if and only if $g(E_\ell) \leq g(E_{\ell'}^k)$ (componentwise) and J_ℓ^2 is empty. We denote by $(E_\ell, E_{\ell'})_{E_{\ell'}^k}$ the arc $(E_\ell, E_{\ell'})$ labeled $E_{\ell'}^k$. We mention that if A is R^* -cyclic, then for all $1 \leq \ell \leq b$ the assumption of Lemma 7.7 holds; this implies that the definition of D is unique. We also define a relation \prec_D on V as follows: $E_{\ell'} \prec_D E_\ell$ if and only if $(E_\ell, E_{\ell'}) \in D$. The relation \prec_D is clearly not symmetric, but transitive. An example of the digraph D with respect to an R^* -cyclic matrix is given in Figure 7.7.

Lemma 7.11 *The computational effort required for the construction of D is bounded by $O(nm\alpha)$.*

Proof. For all $1 \leq \ell \leq b$, let n_ℓ and α_ℓ be the number of rows and nonzero elements of $A_{E_\ell \times f^*(E_\ell)}$. For a given bonsai E_ℓ , the time needed to compute the tree T_I and T in the procedures `Elpath` and `TwoClasses`, respectively, is bounded by $O(n_\ell \alpha_\ell)$, and we need to compute `Elpath`(A, E_ℓ, β) for all $\beta \in f^*(E_\ell)$. Since $n_1 + \dots + n_b \leq n$ and $\alpha_1 + \dots + \alpha_b \leq \alpha$, we deduce that the construction of D requires $O(nm\alpha)$ operations. \blacksquare

If A has an R^* -cyclic or R^* -network representation $G(A)$, then for all $1 \leq \ell \leq b$ we denote by v_ℓ the closest vertex of B_ℓ to the basic subgraph in $G(A)$ with edge index set R^* ; for any node v_i in $G(A)$, we denote by v_i^* the (other) endnode of the basic path joining v_i and the previous subgraph. If A has a $\{1, \rho\}$ -central representation $G(A)$, for any bonsai B_ℓ on the right of $\{e_1, e_\rho\}$ we define v_ℓ and v_ℓ^* as previously. Provided that A has an R^* -cyclic, R^* -network or R^* -central representation $G(A)$, for all $1 \leq \ell \leq b$ and $\beta \in S^*$, we distinguish four cases (see Figures 7.8 and 7.9).

- a_0) The nonbasic edge f_β does not generate any B_ℓ -path.
- a_1) The nonbasic edge f_β generates exactly one B_ℓ -path and $A_{ij} = 1$ for all $i \in E_\ell \cap s(A_{\bullet j})$.
- a_2) The nonbasic edge f_β generates exactly one B_ℓ -path and $A_{ij} = 2$ for all $i \in E_\ell \cap s(A_{\bullet j})$.
- a_3) The nonbasic edge f_β generates two distinct B_ℓ -paths.

Proposition 7.12 *Suppose that A has an R^* -cyclic, R^* -central or R^* -network representation $G(A)$. Let $1 \leq \ell \leq b$ and $\beta \in S^*$. If one faces case a_0 (respectively, a_1 , a_2 and a_3), then the following respective properties hold.*

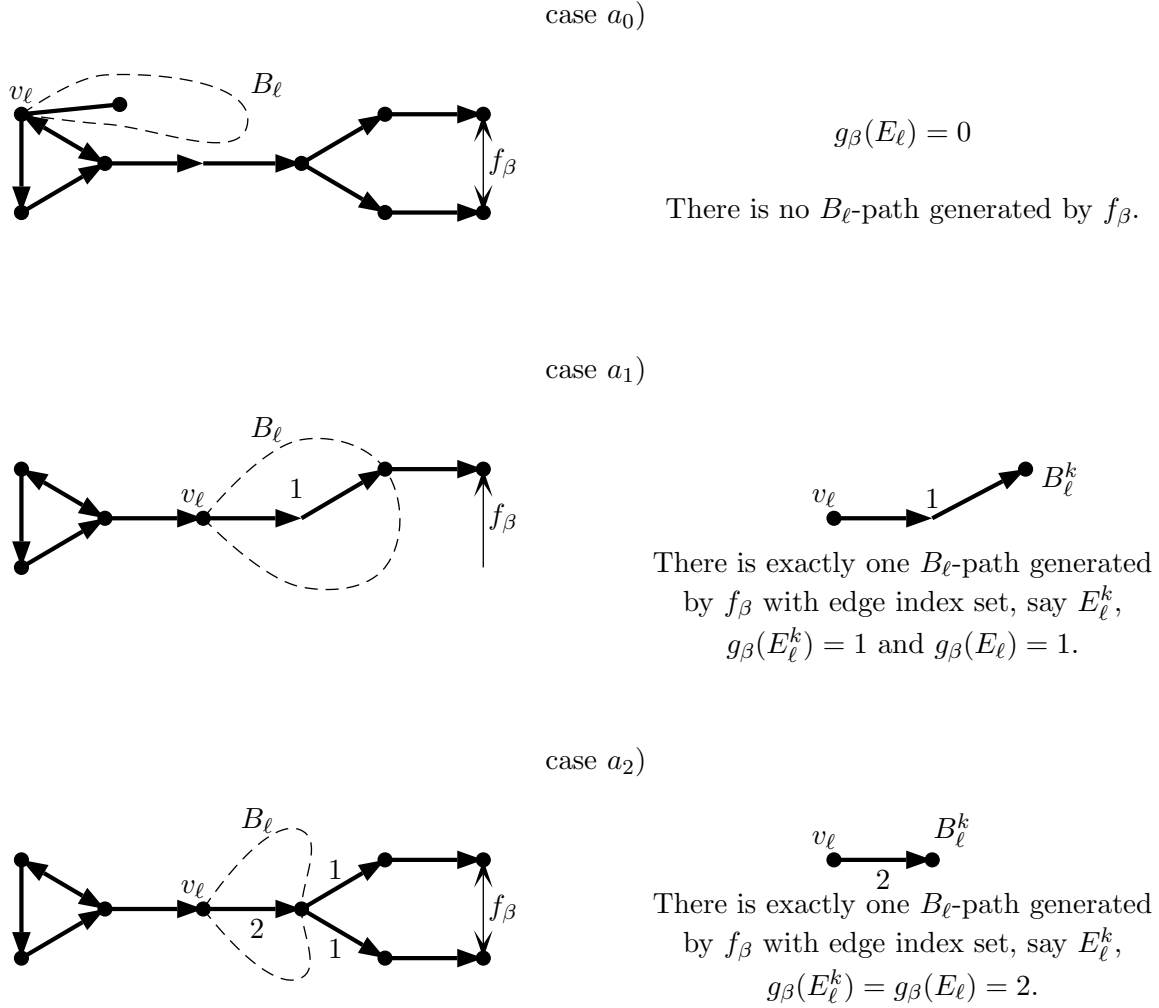


Figure 7.8: A description of all possible B_ℓ -paths generated by a nonbasic edge f_β and the corresponding E_ℓ -paths generated by $A_{\bullet\beta}$ in cases a_0 , a_1 and a_2 . (A number close to an edge corresponds to the weight of the corresponding edge in the fundamental circuit of f_β .)

$$a_0) \quad g_\beta(E_\ell) = 0.$$

$$a_1) \quad E_\ell^{II}(A_{\bullet\beta}) = \emptyset \text{ and } g_\beta(E_\ell) = g_\beta(E_\ell^I(A_{\bullet\beta})) = 1.$$

$$a_2) \quad E_\ell^I(A_{\bullet\beta}) = E_\ell^{II}(A_{\bullet\beta}) \text{ and } g_\beta(E_\ell) = g_\beta(E_\ell^I(A_{\bullet\beta})) = 2.$$

$$a_3) \quad E_\ell^I(A_{\bullet\beta}) \neq E_\ell^{II}(A_{\bullet\beta}), \quad g_\beta(E_\ell^I(A_{\bullet\beta})) = g_\beta(E_\ell^{II}(A_{\bullet\beta})) = 1 \text{ and } g_\beta(E_\ell) = 2.$$

The converse sense is true.

If A is an R^* -network matrix, then cases a_2 and a_3 do not happen.

Proof. This directly follows from the construction of $E_\ell^I(A_{\bullet\beta})$ and $E_\ell^{II}(A_{\bullet\beta})$, Lemma 7.6 and the definition of g . ■

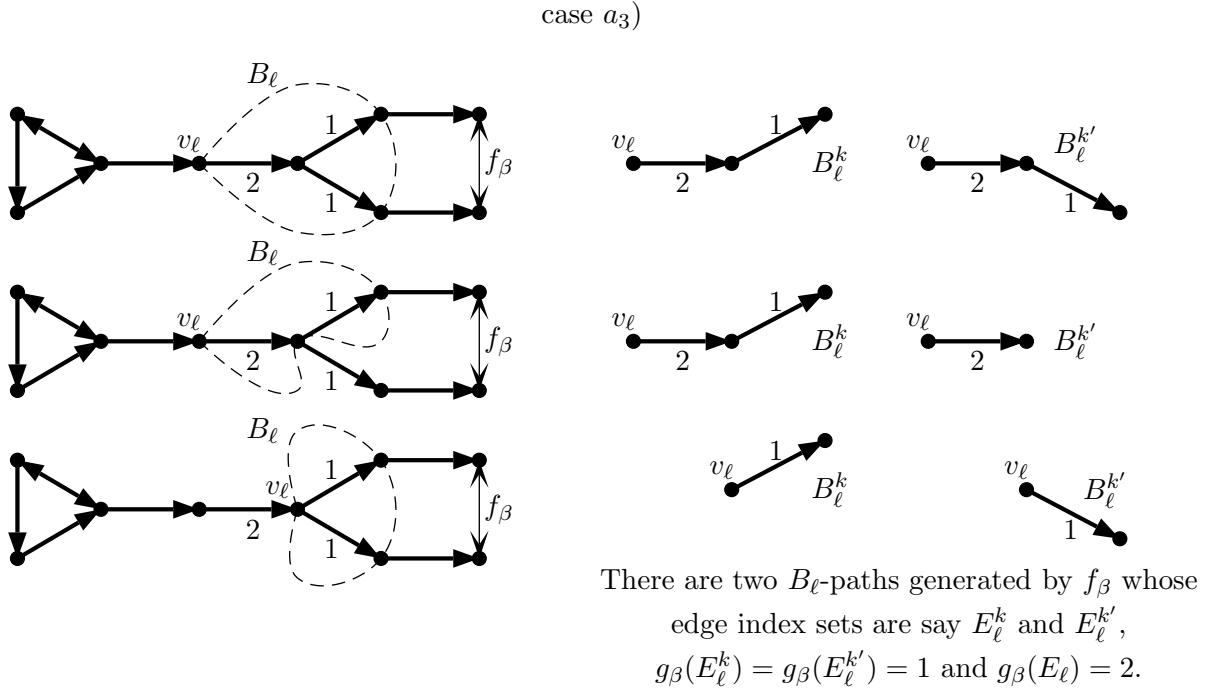


Figure 7.9: A description of all possible B_ℓ -paths generated by a nonbasic edge f_β and the corresponding E_ℓ -paths generated by $A_{\bullet,\beta}$ in case a_3 . (A number close to an edge corresponds to the weight of the corresponding edge in the fundamental circuit of f_β .)

Corollary 7.13 Suppose that A is R^* -cyclic. Let $1 \leq \ell \leq b$ and $\beta \in S^*$ such that $g_\beta(E_\ell) = 2$. Then the fundamental circuit of f_β contains the basic cycle.

Proof. The proof follows from Corollary 3.6, Lemma 4.1 and Proposition 7.12 (see the description of the different types of fundamental circuits at page 41 and Figure 4.4). ■

The next proposition gives a graphical interpretation of an arc in D as illustrated in Figure 7.10. In this figure, the fundamental circuit of the nonbasic edge f_3 intersects the trees B_2 , B_3 and B_4 . We have the inequalities $g_3(E_2) \leq g_3(E_4^1)$ and $g_3(E_4) \leq g_3(E_3^2)$.

Proposition 7.14 Suppose that A has an R^* -cyclic or R^* -network representation $G(A)$. Let $1 \leq \ell, \ell' \leq b$ and q be the basic path in $G(A)$ from v_ℓ to v_ℓ^* . Suppose that $E_{\ell'}$ contains an edge of q . Then $\{i : e_i \in q \cap B_{\ell'}\} = E_{\ell'}^k$ for some $1 \leq k \leq m(\ell')$ and $(E_\ell, E_{\ell'})_{E_{\ell'}^k} \in D$.

Proof. Since A is connected, there exists a column index j in the global connector set $f^*(E_\ell)$ of E_ℓ , such that q is a subpath of a stem issued from the nonbasic edge f_j . Then, $\{i : e_i \in q \cap B_{\ell'}\}$ is an $E_{\ell'}$ -path, say $E_{\ell'}^k$, generated by $A_{\bullet,j}$. By definition of g , we have the inequality $g_j(E_{\ell'}^k) \geq 1$. If $g_j(E_\ell) = 2$ for some column index j , then by Proposition 7.12 one is faced with case a_2 or a_3 with respect to B_ℓ and f_j . It follows that $E_{\ell'}^k$ is the unique $E_{\ell'}$ -path generated by $A_{\bullet,j}$ and $A_{ij} = 2$ for all $i \in E_{\ell'} \cap s(A_{\bullet,j})$. Hence $g_j(E_{\ell'}^k) = 2$. Therefore, $g_j(E_\ell) \leq g_j(E_{\ell'}^k)$. Moreover, since A is nonnegative, if q is leaving v_ℓ , then all B_ℓ -paths of

$G(A)$ are entering v_ℓ . Similarly, if q is entering v_ℓ , then all B_ℓ -paths of $G(A)$ are leaving v_ℓ . Thus by Lemma 7.7 $J_\ell^2 = \emptyset$. We conclude that $(E_\ell, E_{\ell'})_{E_{\ell'}^k} \in D$. ■

Proposition 7.15 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. Let B_ℓ and $B_{\ell'}$ be two bonsais on the right of $\{e_1, e_\rho\}$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$. Then $(E_\ell, E_{\ell'}) \in D$ or $(E_{\ell'}, E_\ell) \in D$.*

Proof. Since $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$, B_ℓ and $B_{\ell'}$ contain each one at least one edge of the fundamental circuit of a nonbasic edge f_j with $j \in f^*(E_\ell) \cap f^*(E_{\ell'})$. Then, the proof is similar to the proof of Proposition 7.14. ■

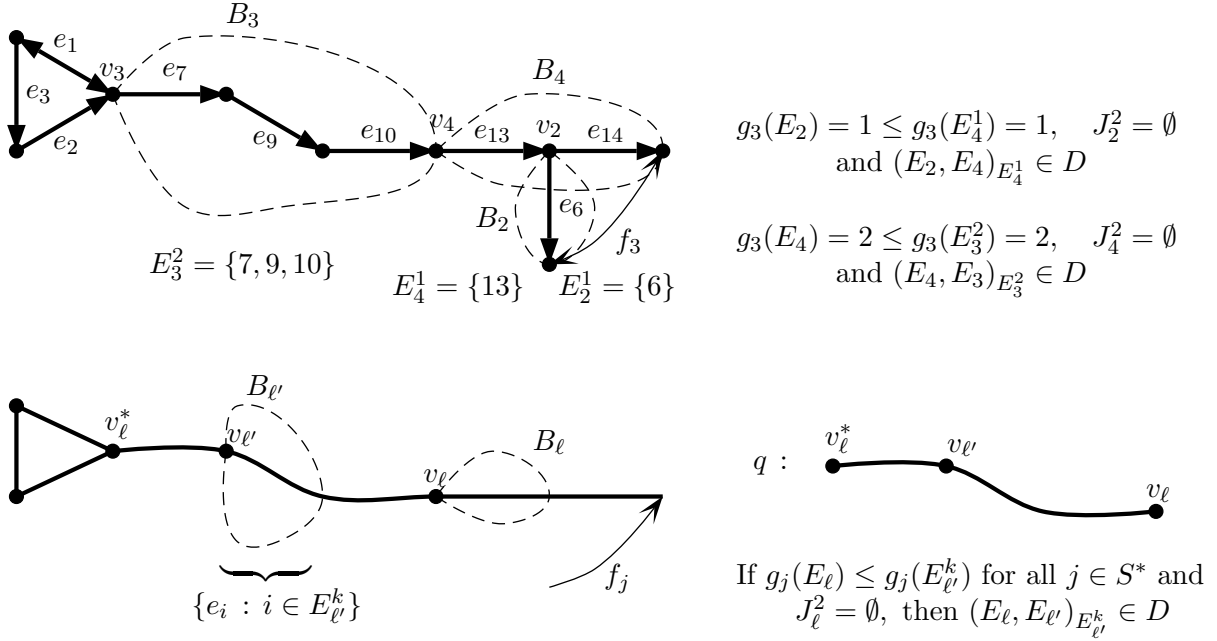


Figure 7.10: The fundamental circuit of f_3 as in Figure 7.2 and an illustration of a stem (issued from some f_j) which intersects a bonsai B_ℓ and thus contains the basic path from v_l to v_l^* .

Finally, to deal with directed cycles in D , we define the following relation. For any $E_\ell, E_{\ell'} \in D$, $E_\ell \sim_s E_{\ell'}$ if and only if E_ℓ and $E_{\ell'}$ are in a same strongly connected component of D . This is clearly an equivalence relation. We state a useful lemma in the case where two bonsais are equivalent under the relation \sim_s .

Lemma 7.16 *Let $E_\ell, E_{\ell'} \in D$ such that $E_\ell \sim_s E_{\ell'}$. Then $g(E_\ell) = g(E_{\ell'})$ and $m(\ell) = m(\ell') = 1$.*

Proof. Since the relation " $E_\ell \prec_D E_{\ell'} \Leftrightarrow (E_\ell, E_{\ell'}) \in D$ " is transitive, it follows that $g(E_\ell^k) = g(E_\ell) = g(E_{\ell'}) = g(E_{\ell'}^{k'})$ for some $1 \leq k \leq m(\ell)$ and $1 \leq k' \leq m(\ell')$.

On the other hand, suppose by contradiction that $m(\ell) \geq 2$. Let h be such that $E_\ell^h \neq E_\ell^k$ and $j \in f^*(E_\ell^h)$. We have $1 \leq g_j(E_\ell^h) \leq g_j(E_\ell) = g_j(E_\ell^k)$. Thus we may assume that $E_\ell^I(A_{\bullet,j}) = E_\ell^h$, $E_\ell^{II}(A_{\bullet,j}) = E_\ell^k$ and we have the inequality $E_\ell^I(A_{\bullet,j}) \neq E_\ell^{II}(A_{\bullet,j})$. By definition of g , it results that $g_j(E_\ell^k) = 1$ and $g_j(E_\ell) = 2$, contradicting the equality $g_j(E_\ell) = g_j(E_\ell^k)$. ■

7.4 A spanning forest in D

Let A be a connected matrix and R^* a row index subset of A . In this section, we deal with the case where A is cyclic or R^* -network, and we study some forests in D deriving from an R^* -cyclic or R^* -central representation of A . Throughout this section, we assume that there is no directed cycle in D .

Suppose that A has an R^* -cyclic or R^* -network representation $G(A)$. By Propositions 7.14, the bidirected graph $G(A)$ induces the forest $T_{G(A)}$ in D defined as follows: for any $B_\ell, B_{\ell'} \subseteq G(A)$, $(E_\ell, E_{\ell'})_{E_{\ell'}^k} \in T_{G(A)}$ if and only if v_ℓ is the endnode ($\neq v_{\ell'}$) of the $B_{\ell'}$ -path of $G(A)$ with edge index set $E_{\ell'}^k$. See Figure 7.11 for an example. Whenever A has a $\{1, \rho\}$ -central representation $G(A)$ whose T is the basic maximal 1-tree, we recall that $G_1(A)$ denotes the component of $T \setminus \{e_1, e_\rho\}$ containing w_ρ (on the right of $\{e_1, e_\rho\}$). If A has a $\{1, \rho\}$ -central representation $G(A)$, then using Proposition 7.15 $T_{G_1(A)}$ is defined with respect to $G_1(A)$ in a same way as $T_{G(A)}$ with respect to $G(A)$. With the intention of capturing the main properties of this forest, we give some more definitions in the next paragraph.

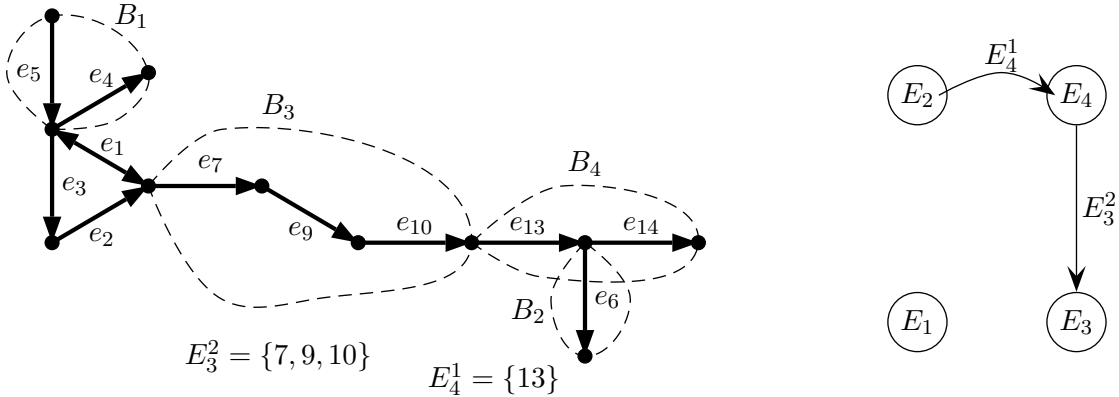
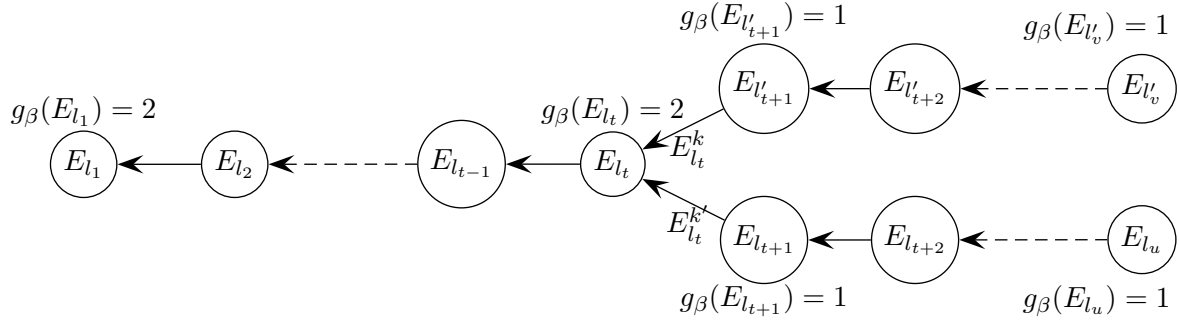


Figure 7.11: A portion of $G(A)$ given in Figure 7.2 and the spanning forest $T_{G(A)}$ of D induced by $G(A)$.

For any $\beta \in S^*$, we define the following objects. A directed path in D is a β -path if for each vertex E_ℓ of the path, we have $g_\beta(E_\ell) = 1$. A β -fork in D is a graph consisting of three directed paths $P_1 = (E_{l_1}, (E_{l_2}, E_{l_1}), E_{l_2}, (E_{l_3}, E_{l_2}), E_{l_3}, \dots, (E_{l_t}, E_{l_{t-1}}), E_{l_t})$, $P_2 = (E_{l_t}, (E_{l_{t+1}}, E_{l_t}), E_{l_{t+1}}, \dots, (E_{l_u}, E_{l_{u-1}}), E_{l_u})$ and $P_3 = (E_{l_t}, (E_{l'_{t+1}}, E_{l_t}), \dots, (E_{l'_v}, E_{l'_{v-1}}), E_{l'_v})$ such that $1 \leq t \leq u, v$ and E_{l_t} is the unique vertex belonging to two of these paths. We have the equalities $g_\beta(E_{l_k}) = 1$ for $t+1 \leq k \leq u$, $g_\beta(E_{l'_k}) = 1$ for $t+1 \leq k \leq v$, $g_\beta(E_{l_k}) = 2$ for $1 \leq k \leq t$. Moreover, if $(E_{l_{t+1}}, E_{l_t})$ and $(E_{l'_{t+1}}, E_{l_t})$ have the same label, say $E_{l_t}^k$, then $g_\beta(E_{l_t}^k) = 2$. If a β -fork is a path, then it is called *simple*. If a β -fork is simple (or equivalently $P_2 = (E_{l_t})$ or $P_3 = (E_{l_t})$), then the node E_{l_t} is said to be β -free. A source vertex of a β -path

or a β -fork is also called a β -free vertex. See Figure 7.12.



If $k = k'$, then $g_\beta(E_{l_t}^k) = 2$.

Figure 7.12: An illustration of a β -fork.

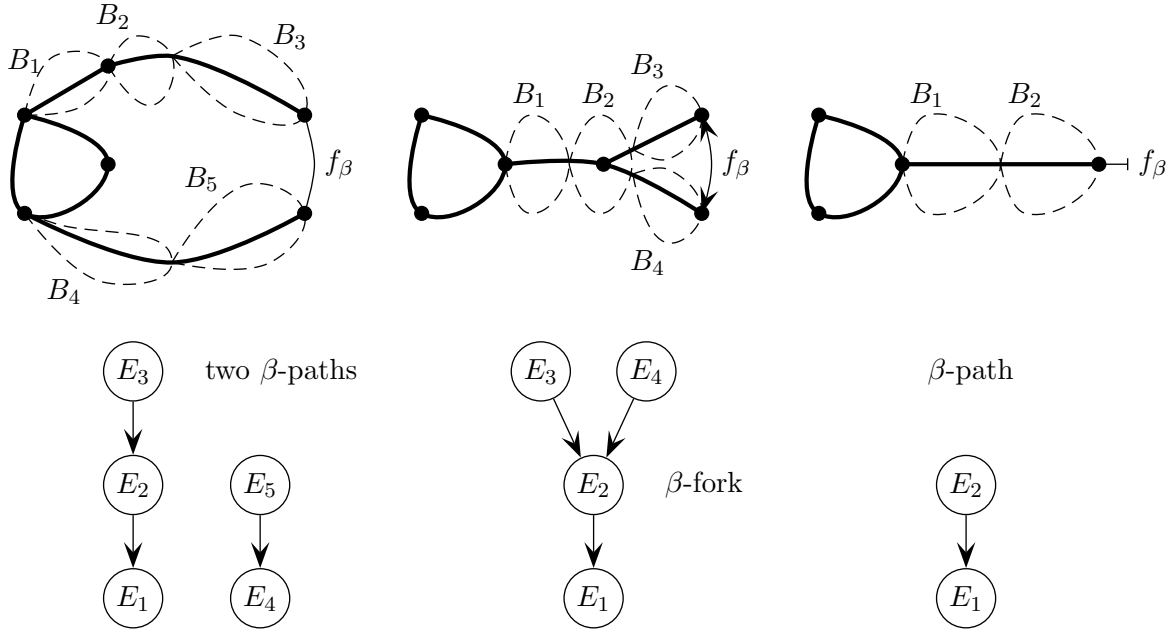


Figure 7.13: An illustration of different configurations of $T_{G(A)}(\{E_\ell : \beta \in f^*(E_\ell)\})$ (at the bottom) with respect to the fundamental circuit of a nonbasic edge f_β with $\beta \in S^*$ (at the top). In each configuration, the basic cycle in $G(A)$ is drawn.

For all $\beta \in S^*$, let $R_\beta = s(A_{\bullet\beta}) \cap R^*$. If $R_\beta \neq R^*$ ($\beta \in S^*$), then the set R_β is called an *interval*. If A has an R^* -cyclic representation $G(A)$, then an interval R_β corresponds to the edge index set of a consistently oriented path in the basic cycle, denoted as p_β , which is called an *interval* in $G(A)$. A forest $T_\Theta = (V', \Theta)$ in D is said to be *feasible* if it has the following properties.

Π_1 : For all $\beta \in S^*$, the subgraph $T_\Theta(\{E_\ell \in V' : \beta \in f^*(E_\ell)\})$ is a β -fork, a β -path or a union of two disjoint β -paths.

Π_2 : For all $\beta \in S^*$, If $s_{\frac{1}{2}}(A_{\bullet\beta}) \neq \emptyset$, then the subgraph $T_{\Theta}(\{E_{\ell} \in V' : \beta \in f^*(E_{\ell})\})$ is a β -path.

Π_3 : For all $\beta, \beta' \in S^*$ such that $R_{\beta} \neq R_{\beta'}$, the subgraph $T_{\Theta}(\{E_{\ell} \in V' : \beta, \beta' \in f^*(E_{\ell}), E_{\ell}^I(A_{\bullet\beta}) \sim_{E_{\ell}} E_{\ell}^I(A_{\bullet\beta'})\})$ is either a β -path or a β' -path.

We denote by $Sink(D)$ the set of sink vertices of D . We observe that the property Π_1 is equivalent to the following.

Π_1^* : For each vertex $E_{\ell} \in V$ and $1 \leq h \leq m(\ell)$, we have
$$\sum_{E_{\ell'} : (E_{\ell'}, E_{\ell})_{E_{\ell}^h} \in \Theta} g(E_{\ell'}) \leq g(E_{\ell}^h).$$

Moreover,
$$\sum_{E_{\ell} \in Sink(D)} g(E_{\ell}) \leq \mathbf{2}_{1 \times s}.$$

On the other hand, in the case where $R^* = \{1, \rho\}$, we define $S_1 = \{j \in S^* : 1 \in s(A_{\bullet j}), \rho \notin s(A_{\bullet j})\}$ and $S_2 = \{j \in S^* : 1 \notin s(A_{\bullet j}), \rho \in s(A_{\bullet j})\}$. A forest $T_{\Theta} = (V', \Theta)$ in D is said to be *right-feasible* if it satisfies Π_1 and the following property.

Π'_2 : For any $\beta \in S_k$ with $k \in \{1, 2\}$, the subgraph $T_{\Theta}(\{E_{\ell} \in V' : \beta \in f^*(E_{\ell})\})$ is a β -path.

Suppose that the matrix A has an R^* -cyclic representation $G(A)$. For any $\beta \in S^*$, the different configurations of the graph $T_{G(A)}(\{E_{\ell} : \beta \in f^*(E_{\ell})\})$ with respect to the fundamental circuit of the nonbasic edge f_{β} are shown in Figure 7.13. For any $\beta, \beta' \in S^*$, denote by v_{β_1} and v_{β_2} (respectively, $v_{\beta'_1}$ and $v_{\beta'_2}$) the endnodes of the nonbasic edge f_{β} (respectively, $f_{\beta'}$). (If f_{β} has one endnode, then $v_{\beta_1} = v_{\beta_2}$.) The pair of nonbasic edges $\{f_{\beta}, f_{\beta'}\}$ is said to be *singular* if $\{v_{\beta_1}^*, v_{\beta_2}^*\} = \{v_{\beta'_1}^*, v_{\beta'_2}^*\}$. Let us see two auxiliary lemmas.

Lemma 7.17 *Suppose that A has a cyclic representation $G(A)$ such that e_1 and e_{ρ} are edges of the basic cycle incident with a common central node. If $1, \rho \in s(A_{\bullet j})$ for some column index j , then the whole basic cycle is contained in the fundamental circuit of f_j .*

Proof. Since A is nonnegative, the proof follows from Corollary 3.6 and Lemma 4.2 (see Figure 4.4). \blacksquare

Lemma 7.18 *Suppose that A has an R^* -cyclic representation $G(A)$. Then the following holds.*

- 1) For any $1 \leq \ell \leq b$ and $\beta \in f^*(E_{\ell})$, if the set $E_{\ell}^{II}(A_{\bullet\beta}) \neq \emptyset$, then $E_{\ell}^I(A_{\bullet\beta}) \sim_{E_{\ell}} E_{\ell}^{II}(A_{\bullet\beta})$.
- 2) For any $1 \leq \ell, \ell' \leq b$ and $1 \leq k \leq m(\ell)$, if $(E_{\ell'}, E_{\ell})_{E_{\ell}^k} \in D$ and $\beta, \beta' \in f^*(E_{\ell'})$, then $\beta, \beta' \in f^*(E_{\ell})$, $E_{\ell'}^I(A_{\bullet\beta}) \sim_{E_{\ell'}} E_{\ell'}^I(A_{\bullet\beta'})$ and $E_{\ell}^I(A_{\bullet\beta}) \sim_{E_{\ell}} E_{\ell}^I(A_{\bullet\beta'})$.

Proof. Using Corollary 3.6 and Lemmas 4.2 and 7.6, one may deduce that the B_{ℓ} -paths with edge index sets $E_{\ell}^I(A_{\bullet\beta})$ and $E_{\ell}^{II}(A_{\bullet\beta})$ are both leaving v_{ℓ} . Then, the proof of part 1 follows from Lemma 7.7.

Now let $1 \leq \ell, \ell' \leq b$ and $1 \leq k \leq m(\ell)$ such that $(E_{\ell'}, E_{\ell})_{E_{\ell}^k} \in D$ and there exist two elements $\beta, \beta' \in f^*(E_{\ell'})$. By definition of an arc in D , the set $J_{\ell'}^2$ is empty. It results that $E_{\ell'}^I(A_{\bullet\beta}) \sim_{E_{\ell'}} E_{\ell'}^I(A_{\bullet\beta'})$. Since the columns $A_{\bullet\beta}$ and $A_{\bullet\beta'}$ generate the same E_{ℓ} -path E_{ℓ}^k , we have the equality $E_{\ell}^i(A_{\bullet\beta}) = E_{\ell}^{i'}(A_{\bullet\beta'})$ for some $i, i' \in \{I, II\}$. Finally, part 1 of the lemma implies that $E_{\ell}^I(A_{\bullet\beta}) \sim_{E_{\ell}} E_{\ell}^I(A_{\bullet\beta'})$. ■

The following Lemma shall be useful in Section 8.2.

Lemma 7.19 *Let $1 \leq \ell, \ell' \leq b$ and suppose that the bonsai matrices N_{ℓ} and $N_{\ell'}$ are network matrices. Then the following holds.*

- 1) *For any $\beta \in f^*(E_{\ell})$, if the set $E_{\ell}^{II}(A_{\bullet\beta}) \neq \emptyset$, then $E_{\ell}^I(A_{\bullet\beta}) \sim_{E_{\ell}} E_{\ell}^{II}(A_{\bullet\beta})$.*
- 2) *If $(E_{\ell'}, E_{\ell})_{E_{\ell}^k} \in D$ and $\beta, \beta' \in f^*(E_{\ell'})$, then $\beta, \beta' \in f^*(E_{\ell})$, $E_{\ell'}^I(A_{\bullet\beta}) \sim_{E_{\ell'}} E_{\ell'}^I(A_{\bullet\beta'})$ and $E_{\ell}^I(A_{\bullet\beta}) \sim_{E_{\ell}} E_{\ell}^I(A_{\bullet\beta'})$.*

Proof. The proof is similar to the proof of Lemma 7.18 by using Lemma 7.9 instead of Lemma 7.7. ■

Now, we are ready to state the main result about the forest $T_{G(A)}$, whenever A has an R^* -cyclic representation $G(A)$.

Theorem 7.20 *Suppose that A has an R^* -cyclic representation $G(A)$. Then $T_{G(A)}$ is a feasible spanning forest of D .*

Proof. The properties Π_1 and Π_2 follow from Proposition 7.12 and the description of the different types of fundamental circuits described at page 41 (see Figures 4.4 and 7.13).

Let us show the property Π_3 . Let $\beta, \beta' \in S^*$ such that $R_{\beta} \neq R_{\beta'}$ and $\{E_{\ell} : \beta, \beta' \in f^*(E_{\ell})\} \neq \emptyset$. We may assume that $R_{\beta'} \neq R^*$, so $R_{\beta'}$ is an interval.

Suppose first that the pair $\{f_{\beta}, f_{\beta'}\}$ is not singular. W.l.o.g, we have $v_{\beta'_1}^* = v_{\beta_1}^*$, $v_{\beta'_2}^* \neq v_{\beta_1}^*$ and $v_{\beta'_2}^* \neq v_{\beta_2}^*$. Therefore, for any E_{ℓ} such that $\beta, \beta' \in f^*(E_{\ell})$, we have $v_{\ell}^* = v_{\beta'_1}^*$. So the graph $T_{G(A)}(\{E_{\ell} : \beta, \beta' \in f^*(E_{\ell})\})$ is a β' -path (it can not be a β' -fork, otherwise $R_{\beta'} = R^*$). Hence, by Lemma 7.18 (part 2), if the subgraph $T_{G(A)}(\{E_{\ell} : \beta, \beta' \in f^*(E_{\ell})\})$ has at least two vertices, then $T_{G(A)}(\{E_{\ell} : \beta, \beta' \in f^*(E_{\ell})\}) = T_{G(A)}(\{E_{\ell} : \beta, \beta' \in f^*(E_{\ell}), E_{\ell}^I(A_{\bullet\beta}) \sim_{E_{\ell}} E_{\ell}^I(A_{\bullet\beta'})\})$. Thus $T_{G(A)}(\{E_{\ell} : \beta, \beta' \in f^*(E_{\ell}), E_{\ell}^I(A_{\bullet\beta}) \sim_{E_{\ell}} E_{\ell}^I(A_{\bullet\beta'})\})$ is a β' -path.

Now suppose that the pair $\{f_{\beta}, f_{\beta'}\}$ is singular. It follows that $R^* = R_{\beta} \uplus R_{\beta'}$. Recall that w_1, \dots, w_{ρ} are the vertices of the basic cycle, $e_1 = [w_1, w_{\rho}]$ and $e_i =]w_{i-1}, w_i]$ for $i = 2, \dots, \rho$. By Lemma 7.17, we may assume that $1 \in R_{\beta} \setminus R_{\beta'}$ and $\rho \in R_{\beta'} \setminus R_{\beta}$. Let $k = \max\{i : e_i \in p_{\beta}\}$. Clearly, $k < \rho$, e_k ($\in p_{\beta}$) enters w_k and $]w_k, w_{k+1}] \in p_{\beta'}$. If there exists a bonsai B_{ℓ} such that $v_{\ell} = w_k$ and $\beta, \beta' \in f^*(E_{\ell})$, then the B_{ℓ} -paths generated by f_{β} and $f_{\beta'}$ leave and enter v_{ℓ} , respectively. Thus, by Lemma 7.7, we have that $E_{\ell}^I(A_{\bullet\beta}) \sim_{E_{\ell}} E_{\ell}^I(A_{\bullet\beta'})$. Hence, by Lemma 7.18, it does not exist an edge $(E_{\ell'}, E_{\ell}) \in D$ such that $\beta, \beta' \in f^*(E_{\ell'})$.

On the other hand, if there is a bonsai B_{ℓ} such that $v_{\ell}^* = w_{\rho}$ and $\beta, \beta' \in f^*(E_{\ell})$, then the B_{ℓ} -paths generated by f_{β} and $f_{\beta'}$ are both leaving v_{ℓ} . Then by Lemma 7.7 $E_{\ell}^I(A_{\bullet\beta}) \sim_{E_{\ell}} E_{\ell}^I(A_{\bullet\beta'})$. Therefore $T_{G(A)}(\{E_{\ell} : \beta, \beta' \in f^*(E_{\ell}), E_{\ell}^I(A_{\bullet\beta}) \sim_{E_{\ell}} E_{\ell}^I(A_{\bullet\beta'})\})$ is a β -path and a β' -path. ■

Eventually, we can provide a lemma analogous to Lemma 7.18 in the case where A has a $\{1, \rho\}$ -central representation $G(A)$, by restricting ourselves to the set of bonsais on the right of $\{e_1, e_\rho\}$. Then, we can prove the second main theorem.

Theorem 7.21 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. Then $T_{G_1(A)}$ is a right-feasible forest in D .*

Proof. The property Π'_2 is simple to show and the remaining part of the proof is similar to the proof of Theorem 7.20. ■

7.5 The procedure Forest

Let A be a matrix with entries 0, 1, 2 or $\frac{1}{2}$ and R^* a row index subset of A . Let $D = (V, \Upsilon)$ be a digraph as constructed in Section 7.3 and $D' = (V', \Upsilon') \subseteq D$ an induced subgraph without any directed cycle. In this section, we describe a procedure called Forest taking A , D' as well as a subset V_0 of V' as input, and computing a spanning forest of D' , when it does not stop.

Let V'_0 denote the set of $E_\ell \in V'$ such that E_ℓ is a sink vertex of D' , or there exists $\beta \in f^*(E_\ell)$ such that $s_{\frac{1}{2}}(A_{\bullet\beta}) \neq \emptyset$, or there exist $\beta, \beta' \in f^*(E_\ell)$ such that R_β and $R_{\beta'}$ are not equal ($R_\beta \neq R_{\beta'}$) and $E_\ell^I(A_{\bullet\beta}) \sim_{E_\ell} E_\ell^I(A_{\bullet\beta'})$.

On the other hand, given a row index $\rho \neq 1$ such that $R^* = \{1, \rho\}$, we define $S_1 = \{j \in S^* : 1 \in s(A_{\bullet j}), \rho \notin s(A_{\bullet j})\}$, $S_2 = \{j \in S^* : 1 \notin s(A_{\bullet j}), \rho \in s(A_{\bullet j})\}$ and let V''_0 denote the set of $E_\ell \in V'$ such that E_ℓ is a sink vertex of D' or $f^*(E_\ell) \cap (S_1 \cup S_2) \neq \emptyset$. We will prove the following.

Theorem 7.22 *The procedure Forest with input A , D' and $V_0 = V'_0$ outputs a feasible spanning forest of D' if and only if one exists.*

Theorem 7.23 *Suppose $R^* = \{1, \rho\}$. The procedure Forest with input A , D' and $V_0 = V''_0$ outputs a right-feasible spanning forest of D' if and only if one exists.*

The construction of a spanning forest of D' can be reduced to a sequence of 2-SAT problems. Assume that vertex sets V_0, \dots, V_{k-1} in V' and a forest $T_\Theta = (U = \cup_{j=0}^{k-1} V_j, \Theta)$ have been constructed, and $V_k \subseteq V' \setminus U$ for some $k \geq 1$. We construct an instance I_k of 2-SAT.

For any $E_\ell \in V_k$, $E_u \in U$ and $1 \leq h \leq m(u)$, an arc $(E_\ell, E_u)_{E_u^h} \in \Upsilon'$ is said to be *legal* if and only if we have the inequality

$$g(E_\ell) + \sum_{E_{u'} : (E_{u'}, E_u)_{E_u^h} \in \Theta} g(E_{u'}) \leq g(E_u^h).$$

For all $E_\ell \in V_k$, if there exists a legal arc $(E_\ell, E_u) \in \Upsilon'$ labeled E_u^h , we define the variable $X_{E_u^h}^\ell$. The variable $X_{E_u^h}^\ell$ gets a true value if and only if we add the arc $(E_\ell, E_u)_{E_u^h}$ in Θ . Now let us see the clauses. If there exist two legal arcs $(E_\ell, E_u)_{E_u^h}, (E_{\ell'}, E_u)_{E_u^h} \in \Upsilon'$ such that

$$g_\beta(E_\ell) + g_\beta(E_{\ell'}) + \sum_{E_{u'} : (E_{u'}, E_u)_{E_u^h} \in \Theta} g_\beta(E_{u'}) > g_\beta(E_u^h)$$

for some $E_\ell, E_{\ell'} \in V_k$ ($\ell \neq \ell'$), $E_u \in U$ and $\beta \in S^*$, we add in I_k the clauses $X_{E_u}^l \vee X_{E_u}^{\ell'}$ and $\bar{X}_{E_u}^l \vee \bar{X}_{E_u}^{\ell'}$. Thus if these clauses are true, exactly one of the arcs $(E_\ell, E_u)_{E_u}^h$ and $(E_{\ell'}, E_u)_{E_u}^h$ will be in T_Θ . The variable $X_{E_u}^l$ is said to be *associated to the vertex* $E_\ell \in V$.

Let us assume that there are at most two variables associated to each vertex in V_k . If exactly one variable, say $X_{E_u}^l$, is associated to a vertex E_ℓ and there exists some $\beta \in f^*(E_\ell)$ such that $\sum_{E_u \in \text{Sink}(T_\Theta)} g_\beta(E_u) = 2$, we set $X_{E_u}^l = 1$. If there are exactly two variables, say $X_{E_u}^l$ and $X_{E_{u'}}^l$, associated to E_ℓ , then we include in I_k the clauses $X_{E_u}^l \vee X_{E_{u'}}^l$ and $\bar{X}_{E_u}^l \vee \bar{X}_{E_{u'}}^l$. Note as before that if these clauses are true, exactly one of both legal arcs leaving E_ℓ will be in T_Θ .

In the procedure Forest, we use a subroutine called $\text{Spanning}V_0$ which outputs a spanning forest of $D'(V_0)$.

Procedure $\text{Spanning}V_0(D'(V_0))$

Input: An induced subgraph $D'(V_0) \subseteq D'$ where $V_0 \subseteq V'$.

Output: A spanning forest $T_\Theta = (V_0, \Theta)$ of $D'(V_0)$.

- 1) let $\Theta = \emptyset$; $U = \emptyset$; $\tilde{D} = D'(V_0)$, $T_\Theta = (U, \Theta)$;
- 2) **while** $\tilde{D} \neq \emptyset$ **do**
- 3) compute W the set of sink vertices of \tilde{D} ;
 add each legal arc from a node of W to a node of U in Θ ;
 $\tilde{D} = \tilde{D} \setminus W$; $U = U \cup W$;
- endwhile**
 output $T_\Theta = (V_0, \Theta)$;

Procedure Forest(D', V_0)

Input: An induced subgraph $D' \subseteq D$ without any directed cycle and a subset $V_0 \subseteq V'$.

Output: either a spanning forest $T_\Theta = (V', \Theta)$ of D' , or stops.

- 1) call $\text{Spanning}V_0(D'(V_0))$;
- 2) let $k = 1$; $U = V_0$; $\tilde{D} = D' \setminus V_0$; $T_\Theta = (U, \Theta)$;
- 3) **while** $\tilde{D} \neq \emptyset$ **do**
- 4) compute V_k the set of sink vertices of \tilde{D} ;
- 5) if some vertex of V_k has more than two associated variables: STOP;
- 6) compute a truth assignment of I_k , if one exists, otherwise STOP;
 for each true variable, add the corresponding arc in Θ ;
 $\tilde{D} = \tilde{D} \setminus V_k$; $U = U \cup V_k$ and $k = k + 1$;
- endwhile**
 output T_Θ ;

Proof of Theorem 7.22. Suppose that $V_0 = V'_0$ and D' has a feasible spanning forest. Let n be the number of passages in loop 3. For all $0 \leq k \leq n$, let $\Theta_k := \Theta$ as in the procedure Forest after k passages in loop 3 and $T_k = (\cup_{j=0}^k V_j, \Theta_k)$, where $T_0 = (V_0, \Theta_0)$ denotes the forest output by $\text{Spanning}V_0$. Let us prove inductively the following.

statement: (i) For all $0 \leq k \leq n$, there exists a feasible spanning forest T_f in D'

- such that $T_f(\cup_{j=0}^k V_j) = T_k$.
(ii) T_n spans V' .

Let r be the number of passages through loop 2 in the procedure SpanningV0. For $k = 0$, statement (i) follows from Claim 2 below. For $j = 1, \dots, r$ denote by U_j the set U after j passages through loop 2. For proving Claim 2, we use the following.

Claim 1. For all $2 \leq j \leq r$ and any vertex $E_\ell \in U_j$, there exists $E_{\ell'} \in U_{j-1}$ such that $(E_\ell, E_{\ell'}) \in \Upsilon'$. Moreover, for each arc $(E_\ell, E_{\ell'}) \in \Upsilon'$, if $E_\ell \in U_j$ and $E_{\ell'} \in U_{j'}$ with $j, j' \geq 1$, then $j > j'$.

Proof of Claim 1. This is due to step 3 in the procedure SpanningV0. ■

Claim 2. For any feasible spanning forest T_f of D' , $T_f(V_0) = T_0$.

Proof of Claim 2. Let $E_\ell \in V_0 = V'_0$ and T_f be a feasible spanning forest of D' . If E_ℓ is a sink vertex of D' , then clearly E_ℓ is a sink vertex of T_f and T_0 .

Suppose now there exist $\beta, \beta' \in f^*(E_\ell)$ such that $R_\beta \neq R_{\beta'}$ and $E_\ell^I(A_{\bullet\beta}) \sim_{E_\ell} E_\ell^I(A_{\bullet\beta'})$. By definition of V'_0 , the graph $T_f(\{E_{\ell'} \in V_0 : \beta, \beta' \in f^*(E_{\ell'}), E_{\ell'}^I(A_{\bullet\beta}) \sim_{E_{\ell'}} E_{\ell'}^I(A_{\bullet\beta'})\})$ is a β -path. By Claim 1 and since D' has no directed cycle, it follows that $T_f(\{E_{\ell'} \in V_0 : \beta, \beta' \in f^*(E_{\ell'}), E_{\ell'}^I(A_{\bullet\beta}) \sim_{E_{\ell'}} E_{\ell'}^I(A_{\bullet\beta'})\}) = T_0(\{E_{\ell'} \in V_0 : \beta, \beta' \in f^*(E_{\ell'}), E_{\ell'}^I(A_{\bullet\beta}) \sim_{E_{\ell'}} E_{\ell'}^I(A_{\bullet\beta'})\})$. Thus, for any $E_{\ell'} \in V_0$ $1 \leq k \leq m(\ell')$, we have $(E_\ell, E_{\ell'})_{E_{\ell'}^k} \in T_f \Leftrightarrow (E_\ell, E_{\ell'})_{E_{\ell'}^k} \in T_0$. We obtain the same conclusion if $s_{\frac{1}{2}}(A_{\bullet\beta}) \neq \emptyset$ for some $\beta \in f^*(E_\ell)$, because $T_f(\{E_{\ell'} \in V_0 : \beta \in f^*(E_{\ell'})\})$ is a β -path. This completes the proof. ■

Let $k \geq 1$. Suppose that the procedure Forest has constructed a forest T_{k-1} , $V_k \neq \emptyset$, and there exists a feasible spanning forest T_f of D' such that $T_f(\cup_{j=0}^{k-1} V_j) = T_{k-1}$. Since T_f satisfies the property Π_1 and $T_f(\cup_{j=0}^{k-1} V_j) = T_{k-1}$, there are at most two legal arcs leaving each vertex of V_k . So there are at most two variables associated to each vertex of V_k and the procedure Forest does not stop at step 5. Moreover, for any variable $X_{E_u^h}^l$ in I_k , by setting $X_{E_u^h}^l = 1$ if and only if $(E_\ell, E_u)_{E_u^h} \in T_f$, one obtains a truth assignment of I_k . This proves that the procedure Forest computes a forest T_k .

Finally, let us prove that there exists a feasible spanning forest T'_f of D' such that $T'_f(\cup_{j=0}^k V_j) = T_k$. By previous arguments, this implies statement (ii). The analogue of Claim 1 is the following.

Claim 3. For any vertex $E_\ell \in V_j$ $1 \leq j \leq n$ and $0 \leq j' \leq j$, there exists $E_u \in V_{j'}$ such that $(E_\ell, E_u) \in \Upsilon'$. Moreover, for each arc $(E_\ell, E_u) \in \Upsilon'$, if $E_\ell \in V_j$ and $E_u \in V_{j'}$ with $j, j' \geq 0$, then $j > j'$.

Proof of Claim 3. This is due to step 4 in the procedure Forest and the transitivity of the relation \prec_D . ■

Now, let us construct a new forest T'_f such that $T'_f(\cup_{j=0}^k V_j) = T_k$. Add any arc $(E_i, E_{i'})$ of T_f such that $E_i, E_{i'} \in V' \setminus \cup_{j=0}^k V_j$ or $E_i, E_{i'} \in \cup_{j=0}^{k-1} V_j$ and any arc of $\Theta_k \setminus \Theta_{k-1}$ in T'_f .

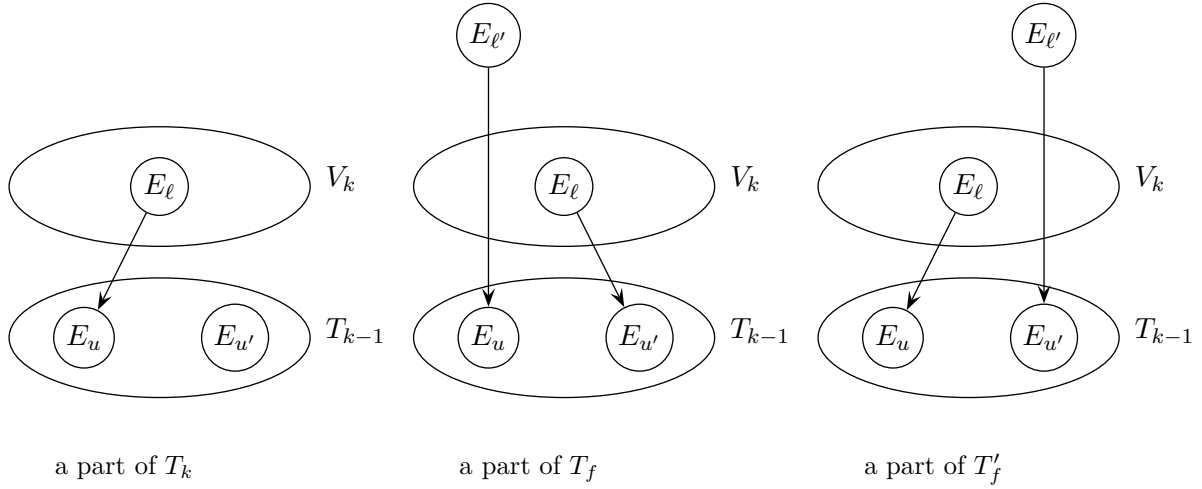


Figure 7.14: An illustration of the proof of Theorem 7.22, in case $(E_{\ell'}, E_{\ell}) \in D'$, $(E_{\ell}, E_u)_{E_u^h} \in T_k$ and $g(E_{\ell}) + g(E_{\ell'}) \not\leq g(E_u^h)$ for some $E_{\ell'} \in V' - \cup_{j=0}^k V_j$, $E_{\ell} \in V_k$ and $E_u \in \cup_{j=0}^{k-1} V_j$.

Then, for any arc $(E_{\ell'}, E_u)_{E_u^h} \in T_f$ such that $E_{\ell} \in V' \setminus \cup_{j=0}^k V_j$ and $E_u \in \cup_{j=0}^{k-1} V_j$, proceed as follows. By Claim 3, there exists $E_{\ell} \in V_k$ such that $(E_{\ell'}, E_{\ell}) \in D'$. If $(E_{\ell}, E_u)_{E_u^h} \notin T_k$ or $g(E_{\ell}) + g(E_{\ell'}) \leq g(E_u^h)$, then add $(E_{\ell'}, E_u)_{E_u^h}$ in T'_f . Otherwise, since T_f satisfies Π_1 , E_{ℓ} is an isolated node in T_f or $(E_{\ell}, E_{u'}^h) \in T_f$ for some vertex $E_{u'} \in \cup_{j=0}^{k-1} V_j$ and $E_u^h \neq E_{u'}^h$. By transitivity of the relation \prec_D , it results that $(E_{\ell'}, E_{u'})_{E_{u'}^h} \in D$, and add $(E_{\ell'}, E_{u'})_{E_{u'}^h}$ in T'_f . Since T_f satisfies Π_1 , for any $\beta \in f^*(E_{\ell'})$, we observe that $T_f(\{E_i \in V' : \beta \in f^*(E_i)\})$ is a non-simple β -fork and $\beta \in f^*(E_{\ell})$, hence $E_{u'}$ has no predecessor E_i in T_f , except $E_i = E_{\ell}$, such that $\beta \in f^*(E_i)$. Thus T'_f satisfies Π_1 . This completes the proof of Theorem 7.22. ■

Proof of Theorem 7.23. The proof uses the same ideas as the proof of Theorem 7.22. ■

Chapter 8

Recognizing R^* -cyclic matrices

Let A be a connected matrix of size $n \times m$ with entries 0, 1, 2, or $\frac{1}{2}$. Let α be the number of nonzero entries of A and R^* a row index subset of A such that $R^* \subseteq s(A_{\bullet j})$ for at least one column index j . In this chapter, we describe a procedure called RCyclic and provide a proof of Theorems 8.1 and 8.2 below. Before reading this chapter, the reader is referred to Chapter 7.

Theorem 8.1 *The matrix A can be tested for having an R^* -cyclic representation by the procedure RCyclic. The running time of this procedure is $O(nm\alpha)$.*

Let us introduce some notations and definitions. We note $S^* = \{j : s(A_{\bullet j}) \cap R^* \neq \emptyset\}$, $\rho = |R^*|$ and $R_j = s(A_{\bullet j}) \cap R^*$, for all $j \in S^*$; the set R_j is called an *interval*. Up to row permutations, we may assume $R^* = \{1, \dots, \rho\}$. Let $D = (V, \Upsilon)$ be a digraph with respect to R^* as computed in Chapter 7. We will see a procedure called Initialization which produces an induced subgraph of D , denoted as D' , having no directed cycle. Further, we will define and construct a set V_c of particular bonsais, called central bonsais, and a matrix $O(R^*)$ called the R^* -open matrix with respect to D' . Let

$$\Lambda_c = \{R_j : j \in f^*(E_\ell) \text{ and } E_\ell \in V_c\}.$$

Whenever A is R^* -cyclic, it will be proved that the poset (Λ_c, \subseteq) can be decomposed into two chains forming an R^* -double chain, a notion that will be defined later. The set V_c is called R^* -compatible if for any $\beta \in S^*$ such that $\sum_{E_\ell \in V_c} g_\beta(E_\ell) \geq 2$, we have the equality $R_\beta = R^*$.

Let D' be a maximal induced subgraph of D having no directed cycle, and $O(R^*)$ the corresponding R^* -open matrix. Let us give two simple necessary conditions for A to be R^* -cyclic that directly follow from Corollary 3.6 and Lemma 4.1. For any $1 \leq j \leq m$, if $s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset$, then $s_{\frac{1}{2}}(A_{\bullet j}) = R^*$ and $s_2(A_{\bullet j}) = \emptyset$; and if $s_2(A_{\bullet j}) \neq \emptyset$, then $s_2(A_{\bullet j}) \cap R^* = \emptyset$ and $R^* \subseteq s(A_{\bullet j})$. Under these two conditions, the following holds.

Theorem 8.2 *The matrix A is R^* -cyclic if and only if the digraph D has a feasible spanning forest, the poset (Λ_c, \subseteq) is an R^* -double chain, the set V_c is R^* -compatible and the R^* -open matrix $O(R^*)$ as well as each bonsai matrix N_ℓ with $E_\ell \in V_c \cup \widehat{\text{Sink}}(D')$ are network matrices.*

Suppose that A has an R^* -cyclic representation $G(A)$. If v_i is a node of $G(A)$, then we note v_i^* the endnode of the basic path from v_i to the basic cycle. Recall that, for all $1 \leq \ell \leq b$,

v_ℓ denotes the closest vertex of B_ℓ to the basic cycle. Moreover, $G(A)$ induces the following spanning forest $T_{G(A)}$ of D : for all $1 \leq \ell, \ell' \leq b$ ($\ell \neq \ell'$), $(E_\ell, E_{\ell'}) \in T_{G(A)}$ if and only if v_ℓ is a node of $B_{\ell'}$ (distinct from $v_{\ell'}$). For all $j \in S^*$, the interval R_j is equal to the edge index set of a consistently oriented path p_j which is called an *interval* in $G(A)$ and lies on the basic cycle.

Before embarking on the proof of Theorems 8.1 and 8.2, Section 8.1 deals with some intuitive notions and graphical ideas on which these are based using an example. Then, in Section 8.2, a formal proof of these theorems is given.

8.1 An informal sketch of a recognition procedure

Let us consider the R^* -cyclic matrix A given in Figure 8.1 where $R^* = \{1, 2, 3\}$. The goal is to construct an R^* -cyclic representation of A , for instance the one given in Figure 8.1, without knowing that such a representation exists. We depict several steps of the recognition procedure RCyclic applied on A , and motivate some definitions in an informal way.

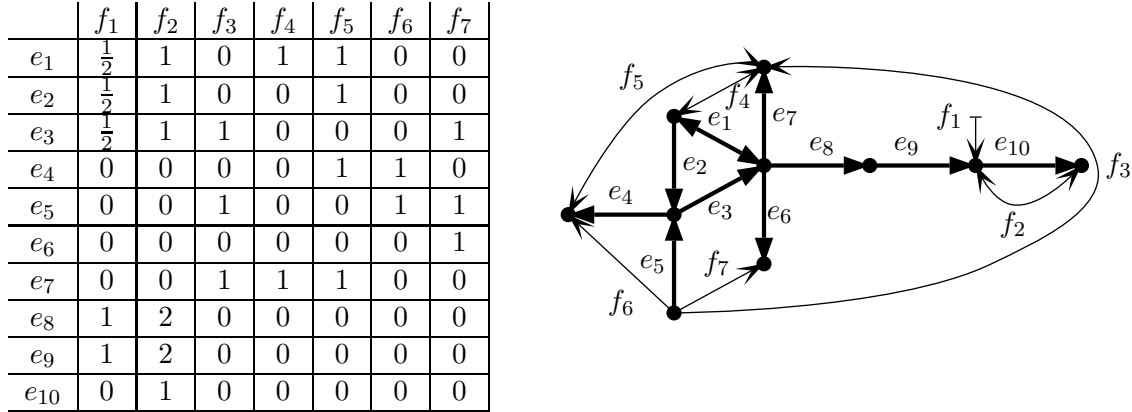


Figure 8.1: A binet matrix A and a $\{1, 2, 3\}$ -cyclic representation $G(A)$ of A .

Suppose that $\overline{R^*}$ has been partitioned into $E_1 = \{4, 5\}$, $E_2 = \{6\}$, $E_3 = \{7\}$, $E_4 = \{8\}$, $E_5 = \{9\}$ and $E_6 = \{10\}$. Let D be a digraph as constructed in Chapter 7 with respect to R^* (see Figure 8.2). In general, let us recall that this construction makes only use of the matrix A , and D is unique, provided that A is R^* -cyclic as mentioned at page 94. We observe that D contains a directed cycle, namely $(E_4, (E_4, E_5), E_5, (E_5, E_4), E_4)$. At first, the procedure RCyclic searches for a maximal induced subgraph in D having no directed cycle, for instance $D \setminus \{E_5\}$. So let $D' = D \setminus \{E_5\}$.

Then, the procedure RCyclic constructs an R^* -cyclic representation of the matrix

$$A' = A_{\cup_{E_\ell \in \text{Sink}(D')} E_\ell \cup R^* \bullet},$$

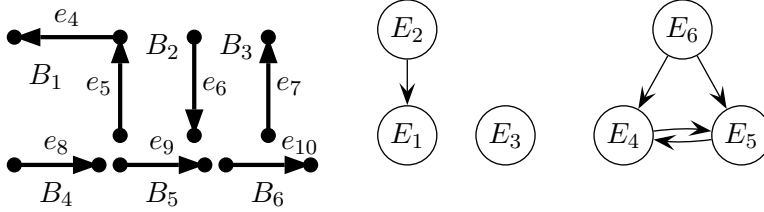


Figure 8.2: the bonsais in $G(A)$ and the digraph D (with respect to $R^* = \{1, 2, 3\}$), where A is given in Figure 8.1.

or determines that A is not R^* -cyclic. How does it proceed? Let us look at the submatrix $A_{R^*, \bullet}^{\frac{1}{2} \rightarrow 1}$ in Figure 8.1. We observe that it is an interval matrix. It will be proved that this follows from the nonnegativity of A and the fact that this matrix is R^* -cyclic. Further, one can extend the matrix $A_{R^*, \bullet}^{\frac{1}{2} \rightarrow 1}$ to a larger one, say $A_{R, \bullet}^{\frac{1}{2} \rightarrow 1}$, so that $R^* \subseteq R$ and $A_{R, \bullet}^{\frac{1}{2} \rightarrow 1}$ is a network matrix. Following this idea, one can try to locate some special row index sets that have to be disjoint from R , otherwise $A_{R, \bullet}^{\frac{1}{2} \rightarrow 1}$ might be a non-network matrix. This motivates a notion of "central" bonsai.

Let E_ℓ be a bonsai. Provided that A has an R^* -cyclic representation $G(A)$, it will be proved that if E_ℓ is central, then v_ℓ corresponds to a central node in $G(A)$, and so all B_ℓ -paths in $G(A)$ are leaving v_ℓ .

Two necessary conditions for E_ℓ to be central are that $J_\ell^2 = \emptyset$ (see Lemmas 7.8 and 7.9) and E_ℓ is a sink vertex of D' . Let us consider the bonsais E_1 and E_3 in $\text{Sink}(D')$. We have that $f^*(E_1) = \{3, 5, 7\}$, $f^*(E_3) = \{3, 4, 5\}$, $R_3 = \{3\}$, $R_4 = \{1\}$, $R_5 = \{1, 2\}$ and $R_7 = \{3\}$. We observe that none of the posets $(\{R_j : j \in f^*(E_1)\}, \subseteq)$ and $(\{R_j : j \in f^*(E_3)\}, \subseteq)$ is a chain. On the other hand, the E_1 -paths (resp., E_3 -paths) generated by the columns of A are $\{4\}$ and $\{5\}$ (resp., $\{7\}$), and $\{4\} \approx_{E_1} \{5\}$ (see Lemma 7.7). Thus $J_1^2 \neq \emptyset$ and $J_3^2 = \emptyset$ (see Lemmas 7.8 and 7.9). Hence E_1 is not central, and we will see that E_3 is central because $E_3 \in \text{Sink}(D')$, $J_3^2 = \emptyset$ and $(\{R_j, j \in f^*(E_3)\}, \subseteq)$ is not a chain. Since $J_1^2 \neq \emptyset$ and $(\{R_j, j \in f^*(E_1)\}, \subseteq)$ is not a chain, provided that A is R^* -cyclic, it can be proved that v_1 is not equal to the central node in any R^* -cyclic representation of A .

Moreover, besides the fact that $E_4 \in \text{Sink}(D')$ and $J_4^2 = \emptyset$, the bonsai E_4 is central for two reasons. The first one is that there exists some $j \in f^*(E_4)$, namely $j = 1$, such that $R_j = R^*$. The second is that there exists some $j \in f^*(E_4)$, namely $j = 2$, such that $g_j(E_4) = 2$. The bonsai E_6 is not central because it is not a sink vertex of D' . Actually, E_3 and E_4 are the only central bonsais in D' . Let

$$R = \cup_{E_\ell \in \text{Sink}(D') \setminus V_c} E_\ell \cup R^* = E_1 \cup R^*.$$

Is the matrix $A_{R, \bullet}^{\frac{1}{2} \rightarrow 1}$ a network matrix? In general, one can show that it is whenever A is R^* -cyclic. Given network representations of $A_{R, \bullet}^{\frac{1}{2} \rightarrow 1}$, N_3 and N_4 , it turns out sometimes that it is not sufficient to construct an R^* -cyclic representation of $A' = A_{\cup_{E_\ell \in \text{Sink}(D')} E_\ell \cup R^*, \bullet}$.

	f_1	f_2	f_6	f_7	f_4^1	f_5^1	f_3^2		
e_1	1	1	0	0	1	1	0	1	1
e_2	1	1	0	0	0	1	0	1	1
e_3	1	1	0	1	0	0	1	1	1
e_4	0	0	1	0	0	1	0	0	0
e_5	0	0	1	1	0	0	1	0	0
e_0	0	0	0	0	1	1	0	0	1
\tilde{e}_3	0	0	0	0	0	0	1	0	1

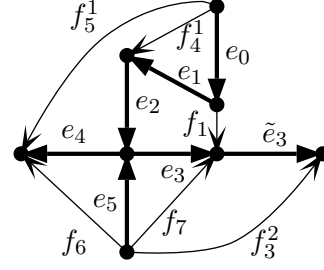


Figure 8.3: the network matrix $O(R^*)$ and a network representation $G(O(R^*))$ of $O(R^*)$ (without some nonbasic edges), where the basic tree is denoted by T .

Let V_c be the set of central bonsais and $\Lambda_c = \{R_j : j \in f^*(E_\ell) \text{ and } E_\ell \in V_c\}$. For our example, $\Lambda_c = \{R^*, \{1\}, \{1, 2\}, \{3\}\}$. Provided that A has an R^* -cyclic representation $G(A)$ as in Figure 8.1, we notice that the intervals in $G(A)$ with edge index set in Λ_c are all incident with the central node, so the poset $(\Lambda_c \setminus \{R^*\}, \subseteq)$ can be split up into two chains, namely $\Lambda_c^1 = (\{\{3\}\}, \subseteq)$ and $\Lambda_c^2 = (\{\{1\}, \{1, 2\}\}, \subseteq)$. The procedure RCyclic produces these two chains or stops, if they do not exist. Then, it constructs a network representation $G(A_{R_\bullet})$ of A_{R_\bullet} such that for $i = 1$ and 2 , every interval in Λ_c^i corresponds to the edge index set of a path, and all these paths have a common endnode, provided that such a representation exists. On this purpose the matrix $O(R^*)$ as given in Figure 8.3 is constructed. Let us remark that $O(R^*)$ has two more rows than A_{R_\bullet} .

Given a basic network representation $G(O(R^*))$ of $O(R^*)$ and, for every $E_\ell \in V_c$, a v_ℓ -rooted network representation B_ℓ of N_ℓ , under certain conditions, the procedure RCyclic constructs a basic R^* -cyclic representation $G(A')$ of the matrix $A' = A_{\cup_{E_\ell \in \text{Sink}(D')} E_\ell \cup R^* \bullet}$ as illustrated in Figure 8.4. This will be accomplished by a subroutine called GASink.

Finally, given a basic R^* -cyclic representation $G(A')$ of A' , a v_ℓ -rooted network representation of N_ℓ for $l = 2, 5$ and 6 , as well as a feasible spanning forest T_Θ of D such that $\text{Sink}(D') \subseteq \text{Sink}(T_\Theta)$, the procedure RCyclic provides a basic R^* -cyclic representation of A as illustrated in Figure 8.5.

8.2 The procedure RCyclic

In this section, we deal with the general framework of the recognition problem, and provide a proof of Theorems 8.1 and 8.2. We describe here the initialization step of the procedure RCyclic. This enables us to focus on an induced subgraph of D without any directed cycle.

Procedure Initialization(A, R^*)

Input: A matrix A and a row index subset R^* of A .

Output: An induced subgraph $D' \subseteq D$ without any directed cycle, or determines that A is not R^* -cyclic.

- 1) **for** every column index j , **do**
 check that if $s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset$, then $s_{\frac{1}{2}}(A_{\bullet j}) = R^*$ and $s_2(A_{\bullet j}) = \emptyset$,
 and if $s_2(A_{\bullet j}) \neq \emptyset$, then $s_2(A_{\bullet j}) \cap R^* = \emptyset$ and $R^* \subseteq s(A_{\bullet j})$;

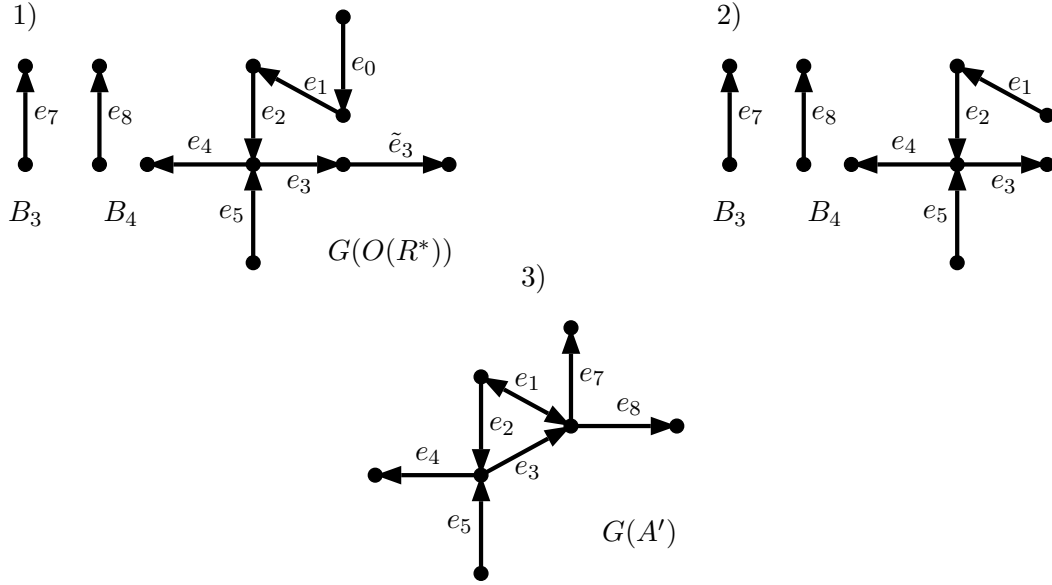


Figure 8.4: Two steps in the construction of a basic $\{1, 2, 3\}$ -cyclic representation $G(A')$ of the matrix A' , where A is given in Figure 8.1. (See pictures from 1 to 3.)

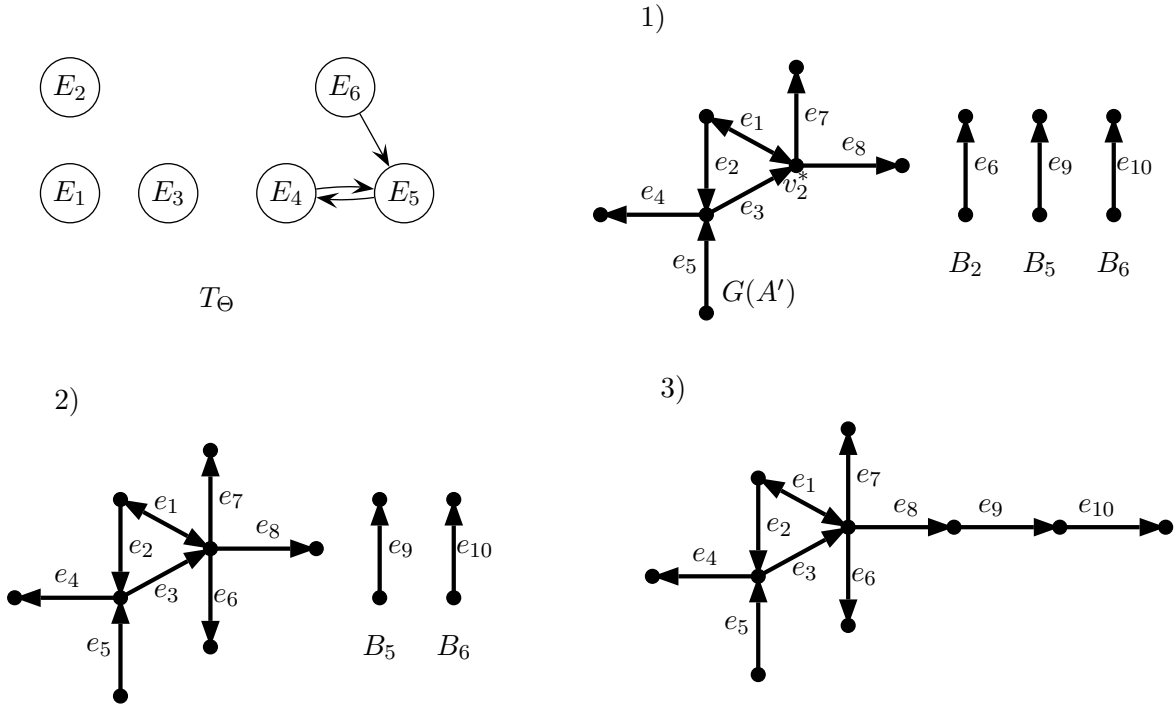


Figure 8.5: A feasible spanning forest T_Θ of D and two steps in the reconstruction of a basic $\{1, 2, 3\}$ -cyclic representation $G(A)$ of A , where A is given in Figure 8.1.

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        otherwise STOP: output that  $A$  is not  $R^*$ -cyclic;
    endfor
2) let  $D' = D$ ;
   while  $D'$  has a directed cycle  $C$ , do
       remove from  $D'$  all vertices of  $C$  except one;
   endwhile
   output  $D'$ ;

```

Suppose that the procedure Initialization has output an induced subgraph D' of D . Let $S_0^* = \{j : R_j = R^*\}$. A spanning forest T_Θ of D is called D' -clean if $\text{Sink}(D') \subseteq \text{Sink}(T_\Theta)$. Provided that A is R^* -cyclic, a D' -clean R^* -cyclic representation $G(A)$ of A verifies the inclusion $\text{Sink}(D') \subseteq \text{Sink}(T_{G(A)})$. A bonsai $E_\ell \in D'$ is designated as a *central* bonsai if $J_\ell^2 = \emptyset$ and E_ℓ is a sink vertex of D' satisfying at least one of the following conditions:

1. The vertex E_ℓ is a β -fork for some $\beta \in f^*(E_\ell)$.
2. The global connector set of E_ℓ intersects S_0^* : $f^*(E_\ell) \cap S_0^* \neq \emptyset$.
3. The poset $(\{R_j : j \in f^*(E_\ell)\}, \subseteq)$ is not a chain.

Let V_c be the set of central bonsais. The set V_c is called R^* -compatible if for any $\beta \in S^*$ such that $\sum_{E_\ell \in V_c} g_\beta(E_\ell) \geq 2$, we have the equality $R_\beta = R^*$.

Throughout this section, whenever A has an R^* -cyclic representation $G(A)$, since $R^* \subseteq s(A_{\bullet j})$ for at least one column index j and A is nonnegative, we may assume that w_1, \dots, w_ρ are the vertices of the basic cycle and $e_1 = [w_1, w_\rho]$, $e_{i+1} = [w_i, w_{i+1}] \in G(A)$ for $i = 1, \dots, \rho - 1$. (If $\rho = 1$, then $e_1 = [w_1, w_1] \in G(A)$.) The following statements are good spices in the cooking of Theorem 8.2.

Proposition 8.3 *If A is R^* -cyclic, then there exists a D' -clean R^* -cyclic representation of A .*

Proof. Suppose that A has an R^* -cyclic representation $G(A)$. Observe that any two bonsais in $\text{Sink}(D')$ are not in a same subpath of $T_{G(A)}$ (otherwise, by transitivity of the relation \prec_D and since D' is an induced subgraph of D , one of both bonsais would not be in $\text{Sink}(D')$, a contradiction). Then, by Lemma 7.16 and construction of D' , it is possible to move if necessary some bonsais in $G(A)$ corresponding to vertices in a directed cycle of D , in order to obtain a D' -clean R^* -cyclic representation of A . ■

Lemma 8.4 *Suppose that A has a D' -clean R^* -cyclic representation $G(A)$. Then for any bonsai $E_\ell \in \text{Sink}(D')$, the node v_ℓ belongs to the basic cycle of $G(A)$.*

Proof. For all $1 \leq \ell \leq b$, if v_ℓ is not a node of the basic cycle, then there exists an edge leaving E_ℓ in the digraph $T_{G(A)}$, contradicting the inclusion $\text{Sink}(D') \subseteq \text{Sink}(T_{G(A)})$. ■

Lemma 8.5 *Suppose that A has an R^* -cyclic representation $G(A)$. If some bonsai $E_\ell \in \text{Sink}(D') \setminus V_c$ is such that $v_\ell = w_\rho$, then $J_\ell^2 = \emptyset$ and the poset $(\{R_j : j \in f^*(E_\ell)\}, \subseteq)$ is a chain.*

Proof. Let $E_\ell \in \text{Sink}(D') \setminus V_c$ such that $v_\ell = w_\rho$. Since the edges e_1 and e_ρ enter the central node w_ρ in $G(A)$ and the matrix A is nonnegative, by Lemma 4.2 it follows that all B_ℓ -paths in $G(A)$ are leaving v_ℓ . Then, by Lemma 7.7, we have that $J_\ell^2 = \emptyset$. By the third condition in the definition of a central bonsai, the proof is done. \blacksquare

Lemma 8.6 *Suppose that A has a D' -clean R^* -cyclic representation $G(A)$. Then for any central bonsai E_ℓ ($1 \leq \ell \leq b$), v_ℓ is equal to the central node in $G(A)$.*

Proof. Let E_ℓ be a sink vertex of D' such that $J_\ell^2 = \emptyset$. If the vertex E_ℓ is a β -fork for some $\beta \in S^*$ or $f^*(E_\ell) \cap S_0^* \neq \emptyset$, since A is nonnegative, the conclusion of the lemma follows from Lemma 4.2 and Corollary 7.13.

Now assume that the poset $(\{R_j : j \in f^*(E_\ell)\}, \subseteq)$ is not a chain. By Lemma 8.4 $v_\ell = v_\ell^*$. Suppose by contradiction that v_ℓ is not a central node, say $v_\ell = w_k$ for some $k < \rho$. Since $J_\ell^2 = \emptyset$, by Lemma 7.7, either all B_ℓ -paths of $G(A)$ enter v_ℓ or leave v_ℓ . Assume that all B_ℓ -paths of $G(A)$ are entering v_ℓ . It results that any interval p_j of $G(A)$ (with $j \in f^*(E_\ell)$) is leaving $v_\ell = w_k$, so p_j contains the edge $]w_k, w_{k+1}]$ and is included in the path $]w_k,]w_k, w_{k+1}], \dots,]w_{\rho-1}, w_\rho], w_\rho]$. Thus the poset $(\{R_j : j \in f^*(E_\ell)\}, \subseteq)$ is a chain, a contradiction. We obtain a similar contradiction if all B_ℓ -paths of $G(A)$ are leaving v_ℓ . \blacksquare

Let Λ and Λ' be two sets whose elements are subsets of R^* . A pair of posets (Λ, \subseteq) and (Λ', \subseteq) is said to be *exclusive* if for all $R \in \Lambda, R' \in \Lambda'$, we have $R \not\subseteq R'$ and $R' \not\subseteq R$. A poset (Λ, \subseteq) is called an *R^* -double chain* if $(\Lambda \setminus \{R^*\}, \subseteq)$ is spanned by an exclusive pair of chains.

Lemma 8.7 *If a poset (Λ, \subseteq) is an R^* -double chain, then a spanning of $(\Lambda \setminus \{R^*\}, \subseteq)$ by an exclusive pair of chains is unique.*

Proof. Clearly, if $(\Lambda \setminus \{R^*\}, \subseteq)$ is a chain, then we can not span it by an exclusive pair of nonempty chains and so there is a unique spanning chain. Suppose by contradiction that there exist four nonempty chains (Λ_1, \subseteq) , (Λ_2, \subseteq) , (Λ_3, \subseteq) and (Λ_4, \subseteq) such that for $i = 1$ and 3 , the pair of chains (Λ_i, \subseteq) and $(\Lambda_{i+1}, \subseteq)$ is exclusive and spans $(\Lambda \setminus \{R^*\}, \subseteq)$, and $\Lambda_3 \neq \Lambda_1, \Lambda_2$. Up to a renumbering of the chains, we may suppose that there exist $R \in \Lambda_3 \cap \Lambda_1$ and $R' \in \Lambda_3 \cap \Lambda_2$. Since (Λ_3, \subseteq) is a chain, it follows that $R \subseteq R'$ or $R' \subseteq R$, which contradicts the fact that the pair of chains (Λ_1, \subseteq) and (Λ_2, \subseteq) is exclusive. \blacksquare

Now let us prove that the poset (Λ_c, \subseteq) is an R^* -double chain, provided that A is R^* -cyclic.

Proposition 8.8 *If A is R^* -cyclic, then the poset (Λ_c, \subseteq) is an R^* -double chain and V_c is R^* -compatible.*

Proof. Suppose that A has an R^* -cyclic representation $G(A)$. By Lemmas 7.17 and 8.6, we deduce that the poset $(\Lambda_c \setminus \{R^*\}, \subseteq)$ is spanned by the exclusive pair of chains $(\{R_j \in \Lambda_c : 1 \in R_j, \rho \notin R_j\}, \subseteq)$ and $(\{R_j \in \Lambda_c : 1 \notin R_j, \rho \in R_j\}, \subseteq)$.

On the other hand, let $E_\ell \in V_c$ such that $g_\beta(E_\ell) = 2$ for some $\beta \in S^*$. By Corollary 7.13 $R_j = R^*$. Similarly, if there exist $E_\ell, E_{\ell'} \in V_c$ such that $g_\beta(E_\ell) = 1$ and $g_\beta(E_{\ell'}) = 1$ for some $\beta \in S^*$, then by Lemma 8.6 $v_\ell = v_{\ell'} = w_\rho$. Following the lines of the proof of Corollary 7.13,

we obtain the same conclusion. ■

By assuming that the poset (Λ_c, \subseteq) is an R^* -double chain, the R^* -open matrix $O(R^*)$ is defined as follows. Let us partition $\Lambda_c \setminus \{R^*\}$ into two subsets Λ_c^1 and Λ_c^2 such that the posets (Λ_c^1, \subseteq) and (Λ_c^2, \subseteq) build up an exclusive pair of chains. Let $F_c^i = \{j : \exists E_\ell \in V_c \text{ s.t. } j \in f^*(E_\ell), R_j \in \Lambda_c^i\}$ for $i = 1$ and 2 , $R = \cup_{E_\ell \in \text{Sink}(D') \setminus V_c} E_\ell \cup R^*$ and $F = f(R) \setminus (F_c^1 \cup F_c^2)$. Then

$$O(R^*) = \begin{bmatrix} A_{R \times F}^{\frac{1}{2} \rightarrow 1} & A_{R \times F_c^1} & A_{R \times F_c^2} & \chi_{R^*}^R & \chi_{R^*}^R \\ \mathbf{0}_{1 \times \mathbf{m}} & \mathbf{1}_{1 \times |F_c^1|} & \mathbf{0}_{1 \times |F_c^2|} & 0 & 1 \\ \mathbf{0}_{1 \times \mathbf{m}} & \mathbf{0}_{1 \times |F_c^1|} & \mathbf{1}_{1 \times |F_c^2|} & 0 & 1 \end{bmatrix},$$

where the rows of $O(R^*)$ except the two last ones are indexed by the elements of the set R . If $O(R^*)$ has a network representation $G(O(R^*))$, then the two last rows of $O(R^*)$ correspond to basic *artificial* edges denoted as e_0 and \tilde{e}_ρ . Let us see the graphical interpretation of $O(R^*)$.

Proposition 8.9 *Suppose that A is R^* -cyclic. Then the R^* -open matrix $O(R^*)$ is a network matrix.*

Proof. By Proposition 8.3, let $G(A)$ be a basic D' -clean R^* -cyclic representation of A . Let $\Lambda^1 = \{R_j : \rho \notin R_j, 1 \in R_j\}$ and $\Lambda^2 = \{R_j : 1 \notin R_j, \rho \in R_j\}$. For $i = 1$ and 2 , let $\mathcal{B}^i = \{E_\ell \in \text{Sink}(D') \setminus V_c : v_\ell = w_\rho, \{R_j, j \in f^*(E_\ell)\} \subseteq \Lambda^i\}$.

For any $E_\ell \in \text{Sink}(D') \setminus V_c$ such that $v_\ell = w_\rho$, we make the following observations. From the definition of a central bonsai and Lemma 8.5, it results that $R_j \neq R^*$ for all $j \in f^*(E_\ell)$ and the poset $(\{R_j : j \in f^*(E_\ell)\}, \subseteq)$ is a chain. Hence, using Lemma 7.17, the set $\{R_j : j \in f^*(E_\ell)\}$ is included in Λ^1 or Λ^2 . Thus

$$\{E_\ell \in \text{Sink}(D') \setminus V_c : v_\ell = w_\rho\} = \mathcal{B}^1 \uplus \mathcal{B}^2. \quad (8.1)$$

Furthermore, by Lemmas 7.17 and 8.6, $\Lambda_c \setminus \{R^*\} \subseteq \Lambda^1 \cup \Lambda^2$. So by Lemma 8.7, we may assume that

$$\Lambda_c^1 = \Lambda_c \cap \Lambda^1 \text{ and } \Lambda_c^2 = \Lambda_c \cap \Lambda^2. \quad (8.2)$$

Since $G(A)$ is D' -clean, by Lemma 8.4 we deduce that R is the edge index set of a connected graph in $G(A)$. Using $G(A)$, we construct a basic network representation T of the matrix $O(R^*)$. First, let us contract all basic edges with index in $\cup_{E_\ell \in \overline{\text{Sink}(D')}} \cup V_c E_\ell$. Then create two new basic edges (and vertices) $e_0 =]\tilde{w}_0, w_0]$ and $\tilde{e}_\rho =]w_\rho, \tilde{w}_\rho]$ and replace the bidirected edge $[w_1, w_\rho]$ by $]w_0, w_1]$. Finally, for every $E_\ell \in \mathcal{B}^1$, let us get the bonsai B_ℓ loose from the central node (by making a copy of $v_\ell = w_\rho$), reverse the orientation of all its basic edges and identify the copy of v_ℓ with w_0 . (See Figure 8.3 for an example.)

Let us prove that T is a basic network representation of $O(R^*)$. To see that each column of $O(R^*)$ is the edge incidence vector of a directed path in T , we state two claims.

Claim 1: For all $j \in F$, the column $[(A_{R \times \{j\}}^{\frac{1}{2} \rightarrow 1})^T \ 0 \ 0]^T$ is the edge incidence vector of a directed path in T .

Claim 2: For all $j \in F_c^1$ (respectively, $j \in F_c^2$), $s(A_{\bullet j}) \cap R$ is the edge index set of a directed path in T whose w_0 (resp., w_ρ) is an endnode.

Proof of Claim 1. Let $j \in F$. If $1, \rho \notin s(A_{\bullet j})$, then by construction of T , up to a reversing of all edges the paths in $G(A)$ and in T with edge index set $s(A_{\bullet j}) \cap R$ are isomorphic. Suppose that $1, \rho \in s(A_{\bullet j})$. By Lemma 7.17 $R_j = R^*$. If j is in the global connector set of a bonsai $E_\ell \in \text{Sink}(D')$, then by Lemma 8.4 $v_\ell = w_\rho$. Hence, by Lemma 8.5 or the definition of a central bonsai $J_\ell^2 = \emptyset$. Therefore $E_\ell \in V_c$, a contradiction. Thus $s(A_{\bullet j}) \cap R = R^*$ is the edge index set of a directed path in T . At last, if exactly one of the indexes 1 or ρ is in $s(A_{\bullet j})$, then by equation (8.1) and construction of T , the proof follows.

Proof of Claim 2. Let $j \in F_c^1$ (the proof is symmetric with $j \in F_c^2$). By definition of F_c^1 , there exists a bonsai $E_\ell \in V_c$ such that $j \in f^*(E_\ell)$ and $R_j \in \Lambda_c^1$. Hence $1 \in s(A_{\bullet j}) \cap R$, $\rho \notin s(A_{\bullet j}) \cap R$ and by Lemma 8.6 $v_\ell = w_\rho$. Since $E_\ell \cap R = \emptyset$, it follows that $s(A_{\bullet j}) \cap R$ is the edge index set of a directed path in $G(A)$ with one endnode equal to w_ρ . Moreover, by (8.2), $R_j \in \Lambda^1$. Then, by construction of T , the proof of the claim follows.

The vectors $[(\chi_{R^*}^R)^T \ 0 \ 0]^T$ and $[(\chi_{R^*}^R)^T \ 1 \ 1]^T$ are clearly edge incidence vectors of directed paths in T . Using Claims 1 and 2, it is easy to conclude the proof of Proposition 8.9. ■

Suppose that the matrix $O(R^*)$ has a basic network representation $G(O(R^*))$. Since $[(\chi_{R^*}^R)^T \ 0 \ 0]^T$ is a column of $O(R^*)$, the set R^* is the edge index set of a directed path denoted as p^* in $G(O(R^*))$. Moreover, since A is connected $(O(R^*))_{R^*}$ is connected, hence R is the edge index set of a tree in $G(O(R^*))$. Thus, if $F_c^1 \neq \emptyset$ and $F_c^2 \neq \emptyset$, since the vectors $[(\chi_{R^*}^R)^T \ 1 \ 1]^T$, $[A_{R \times \{j_1\}}^T \ 1 \ 0]^T$ and $[A_{R \times \{j_2\}}^T \ 0 \ 1]^T$ (for some $j_1 \in F_c^1$, $j_2 \in F_c^2$) are edge incidence vectors of paths in $G(O(R^*))$, we deduce that e_0 is incident with one endnode of p^* and \tilde{e}_ρ with the other one. We obtain the same conclusion that whenever $F_c^1 = \emptyset$ or $F_c^2 = \emptyset$. By reversing the orientation of all edges in $G(O(R^*))$ if necessary, we may note

$$p^* =]w_0,]w_0, w_1], \dots,]w_{\rho-1}, w_\rho], w_\rho],$$

$e_0 =]\tilde{w}_0, w_0]$ and $\tilde{e}_\rho =]w_\rho, \tilde{w}_\rho]$. For ease of notation, a basic subtree in $G(O(R^*))$ with edge index set equal to E_ℓ for some $1 \leq \ell \leq b$ is called a bonsai B_ℓ , and v_ℓ denotes the closest node in B_ℓ to the path p^* .

At this point, we are ready to describe the main subroutine of the procedure RCyclic. Let

$$A' = A_{\cup_{E_\ell \in \text{Sink}(D')} E_\ell \cup R^* \bullet}.$$

In the following procedure, for each $E_\ell \in V_c$, if N_ℓ has a v_ℓ -rooted network representation B_ℓ , then up to a reversing of all edges in B_ℓ , we assume that all B_ℓ -paths in B_ℓ leave v_ℓ (since $J_\ell^2 = \emptyset$, this is justified by Lemma 7.9).

Procedure GASink(A, R^*)

Input: A matrix A and a row index subset R^* of A such that $R^* \subseteq s(A_{\bullet j})$

for at least one column index j .

Output: Either a basic R^* -cyclic representation $G(A')$ of $A' = A_{\cup_{E_\ell \in \text{Sink}(D')} E_\ell \cup R^* \bullet}$ and $D' \subseteq D$, or determines that A is not R^* -cyclic;

- 1) call **Initialization**(A, R^*) outputting a subgraph $D' \subseteq D$, or the fact that A is not R^* -cyclic.
- 2) compute, if they exist, $O(R^*)$, a basic network representation $G(O(R^*))$ of $O(R^*)$ and a v_ℓ -rooted network representation B_ℓ of N_ℓ for every $E_\ell \in V_c$, otherwise STOP: output that A is not R^* -cyclic;
- 3) check that V_c is R^* -compatible, otherwise STOP: output that A is not R^* -cyclic;
- 4) check that $\text{Sink}(D')$ is feasible, otherwise STOP: output that A is not R^* -cyclic;
- 5) delete e_0 , \tilde{e}_ρ , \tilde{w}_0 and \tilde{w}_ρ ; replace the edge $]w_0, w_1]$ by $[w_0, w_1]$; reverse the orientation of all edges of the bonsais connected at w_0 ; identify w_0 with w_ρ , and for each $E_\ell \in V_c$ identify v_ℓ with w_ρ ; output the resulting 1-tree and $D' \subseteq D$;

Let us show the correctness of the procedure GASink.

Proposition 8.10 *The output of the procedure GASink is correct.*

Proof. Suppose that A is R^* -cyclic. By Corollary 3.6, Lemma 4.1 and the description of the different types of fundamental circuits given at page 41 (see Figure 4.4), the subroutine Initialization does not stop at its step 1 and produces an induced subgraph $D' \subseteq D$ without any directed cycle by construction. By Proposition 8.8, the R^* -open matrix $O(R^*)$ is well defined. Furthermore, by Proposition 8.9 and Lemma 7.8 (respectively, Proposition 8.8 and Theorem 7.20), the procedure GASink does not stop at step 2 (respectively, step 3 and 4). Finally, by construction, it outputs a 1-tree.

Now suppose that the procedure outputs a 1-tree denoted as $G(A')$. One needs to show that each column in A' can be interpreted as a binet matrix whose $G(A')$ is a basic binet representation. Let j_0 be an index of a nonzero column in A' .

If $s(A'_{\bullet j_0}) \cap R^* = \emptyset$, then since $s(A'_{\bullet j_0}) \subseteq E_\ell$ for some bonsai $E_\ell \in \text{Sink}(D')$ and B_ℓ is a v_ℓ -rooted network representation of N_ℓ it follows that $A'_{\bullet j_0}$ is the edge incidence vector of a directed path in $G(A')$. Now assume that $j_0 \in S^*$. Since $\text{Sink}(D')$ is feasible, j_0 is in the global connector set of at most two bonsais. If j_0 is not in the global connector set of a central bonsai, then $[A'_{R \times \{j_0\}}^T \ 0 \ 0]^T$ is the edge incidence vector of a directed path in $G(O(R^*))$. So by construction $s(A'_{\bullet j_0}) \cap R = s_1(A'_{\bullet j_0}) \cap R$ and $s(A'_{\bullet j_0}) \cap R$ is the edge index set of a consistently oriented path or a pathcycle.

Suppose that j_0 is in the global connector set of a central bonsai, say E_{l_c} , and a non-central one, say E_ℓ in $\text{Sink}(D')$. Suppose by contradiction that $R_{j_0} = R^*$. Since $s(A'_{\bullet j_0}) = E_\ell^k \cup R^*$ for some $1 \leq k \leq m(\ell)$, it follows that $v_\ell = w_0$ or $v_\ell = w_\rho$ in $G(O(R^*))$; this implies that all paths in $G(O(R^*))$ with edge index set equal to $E_\ell^1, \dots, E_\ell^{m(\ell)}$ leave v_ℓ or they all enter v_ℓ . Then, the bonsai B_ℓ of $G(O(R^*))$ with one more edge entering v_ℓ is a v_ℓ -rooted network representation of N_ℓ , so by Lemma 7.7 $J_\ell^2 = \emptyset$. Hence E_ℓ is central, a contradiction. Therefore $R_{j_0} \neq R^*$. It follows that j_0 belongs to F_c^1 or F_c^2 . Assume $j_0 \in F_c^1$ (if $j_0 \in F_c^2$, then the proof is similar). Since $[A'_{R \times \{j_0\}}^T \ 1 \ 0]^T$ is the edge incidence vector of a directed path in $G(O(R^*))$ containing $e_0 =]\tilde{w}_0, w_0]$ and a subpath of $p^* =]w_0,]w_o, w_1], \dots, w_\rho]$, this implies that v_ℓ is

an inner node of the path p^* ($v_\ell \neq w_0$ and $v_\ell \neq w_\rho$). Moreover, since V_c is R^* -compatible, $g_{j_0}(E_{l_c}) = 1$. So $A'_{\bullet j_0}$ is the incidence vector of a consistently oriented path in $G(A')$.

If j_0 is in the global connector set of one or two central bonsais, with similar arguments one may prove that $A'_{\bullet j_0}$ is the edge index set of some basic fundamental circuit in $G(A')$. This completes the proof. \blacksquare

The following lemma will justify a way of connecting some bonsais to a basic R^* -cyclic representation of A' whenever A is R^* -cyclic.

Lemma 8.11 *Suppose that A' has a basic R^* -cyclic representation $G(A')$. Let T_Θ be a D' -clean feasible spanning forest of D and $E_\ell \in \text{Sink}(T_\Theta) \setminus \text{Sink}(D')$ such that N_ℓ is a network matrix. Then $s(A'_{\bullet j}) = s(A'_{\bullet j'})$ for all $j, j' \in f^*(E_\ell)$. Moreover, for any $j \in f^*(E_\ell)$, $s(A'_{\bullet j})$ is the edge index set of a pathcycle, or a consistently oriented path with exactly one endnode lying on the basic cycle of $G(A')$.*

Proof. By construction of D' and using the transitivity of the relation \prec_D , there exists a sink vertex of D' , say $E_{\ell'}$, such that $(E_\ell, E_{\ell'}) \in D$. By the property Π_1 or Π_1^* satisfied by T_Θ , it follows that $g_\beta(E_\ell) = 1$, $g_\beta(E_{\ell'}) = 1$ and $g_\beta(E_{\ell''}) = 0$ for any $\beta \in f^*(E_\ell)$ and $E_{\ell''} \in \text{Sink}(T_\Theta) \setminus \{E_\ell, E_{\ell'}\}$. Suppose that there exist two indexes $\beta, \beta' \in f^*(E_\ell)$. By Lemma 7.19 (part 2) this implies that $E_{\ell'}^I(A_{\bullet \beta}) \sim_{E_{\ell'}} E_{\ell'}^I(A_{\bullet \beta'})$ and $E_\ell^I(A_{\bullet \beta}) \sim_{E_\ell} E_\ell^I(A_{\bullet \beta'})$, hence the graph $T_\Theta(\{E_u \in \text{Sink}(T_\Theta) : \beta, \beta' \in f^*(E_u), E_u^I(A_{\bullet \beta}) \sim_{E_u} E_u^I(A_{\bullet \beta'})\})$ is neither a β -path nor a β' -path. As T_Θ is feasible, T_Θ satisfies in particular property Π_3 and we deduce that $R_\beta = R_{\beta'}$. This implies Lemma 8.11. \blacksquare

By Lemma 8.11, if A' has a basic R^* -cyclic representation $G(A')$ and D some D' -clean feasible spanning forest T_Θ , then for any $E_\ell \in \text{Sink}(T_\Theta) \setminus \text{Sink}(D')$, we define a vertex v_ℓ^* in $G(A')$ as follows. For any $j \in f^*(E_\ell)$, if $s(A_{\bullet j}) \cap R^* \neq R^*$, then v_ℓ^* is equal to the endnode of the path with edge index set $s(A_{\bullet j}) \cap E$ lying on the basic cycle, otherwise v_ℓ^* is the central node.

Finally, we provide the procedure RCyclic for testing whether A is R^* -cyclic or not. Let V'_0 denote the set of $E_\ell \in V'$ such that E_ℓ is a sink vertex of D' , or there exists $\beta \in f^*(E_\ell)$ such that $s_{\frac{1}{2}}(A_{\bullet \beta}) \neq \emptyset$, or there exist $\beta, \beta' \in f^*(E_\ell)$ such that the intervals R_β and $R_{\beta'}$ are not equal ($R_\beta \neq R_{\beta'}$) and $E_\ell^I(A_{\bullet \beta}) \sim_{E_\ell} E_\ell^I(A_{\bullet \beta'})$. We use the subroutine Forest described in Section 7.5 taking A , D' and a subset $V_0 \subseteq V'$ as input and outputting, for $V_0 = V'_0$, a feasible spanning forest of D' whenever A is R^* -cyclic.

Procedure RCyclic(A, R^*)

Input: A matrix A and a row index subset R^* of A such that $R^* \subseteq s(A_{\bullet j})$

for at least one column index j .

Output: Either a basic R^* -cyclic representation $G(A)$ of A ,
or determines that none exists.

- 1) call **Gasink**(A, R^*) outputting a basic R^* -cyclic representation $G(A')$ of some submatrix A' of A and $D' \subseteq D$, or the fact that A is not R^* -cyclic;
- 2) call **Forest**($D', V_0 = V'_0$) and check that the output of **Forest** is a feasible forest, otherwise STOP: output that A is not R^* -cyclic;
compute a D' -clean feasible spanning forest T_Θ of D ;

- 3) for every $E_\ell \in V(D) \setminus V(D')$, compute a v_ℓ -rooted network representation B_ℓ of N_ℓ , if one exists; otherwise STOP: output that A is not R^* -cyclic;
- 4) **for** each $E_\ell \in \text{Sink}(T_\Theta) \setminus \text{Sink}(D')$, **do**
 identify the node v_ℓ with v_ℓ^* and orient the edges of B_ℓ so that for any $1 \leq k \leq m(\ell)$
 and $\beta \in f^*(E_\ell)$, $E_\ell^k \cup R_\beta$ is the edge index set of a consistently oriented path;
 endfor
- 5) **for** any $E_{\ell'} \in T_\Theta \setminus \text{Sink}(T_\Theta)$, **do**
 let $(E_{\ell'}, E_\ell)_{E_\ell^k} \in T_\Theta$ and identify $v_{\ell'}$ with the endnode ($\neq v_\ell$) of the B_ℓ -path
 with edge index set E_ℓ^k , and orient the edges of $B_{\ell'}$ so that for all $1 \leq j \leq m(\ell')$,
 $E_{\ell'}^j \cup E_\ell^k$ is the edge index set of a directed path;
 endfor
 output the obtained 1-tree;

Proof of Theorems 8.1 and 8.2. The proof of the "only if" part of Theorem 8.2 follows from Propositions 8.3, 8.8 and 8.9 and Lemma 7.8.

Let us show the correctness of the procedure RCyclic. Suppose that A is R^* -cyclic. By Proposition 8.10, the subroutine GASink produces a basic R^* -cyclic representation $G(A')$ of A' . By Theorem 7.22, the subroutine Forest outputs a feasible spanning forest of D' . Then, by Lemma 7.16, it is easy to compute a D' -clean feasible spanning forest of D . By Lemma 7.8, in step 3 a v_ℓ -rooted network representation of the bonsai matrix N_ℓ is computed, for every $E_\ell \in V(D) \setminus V(D')$. By Lemma 8.11 it follows that for any $E_\ell \in \text{Sink}(T_\Theta) \setminus \text{Sink}(D')$ the vertex v_ℓ^* is well defined. At step 4 and 5, for any $E_\ell \in T_\Theta$ and $1 \leq k \leq m(\ell)$, E_ℓ^k is the edge index set of a path in B_ℓ by Lemma 7.9. Finally, by construction, the procedure RCyclic outputs a 1-tree $G(A)$.

Now suppose that the the procedure RCyclic outputs a 1-tree $G(A)$. One need to show that $G(A)$ is a basic R^* -cyclic representation of A . At the end of step 1, by Proposition 8.10 we have a basic R^* -cyclic representation of A' . Then, using Lemmas 8.11 and 7.9 and the fact that T_Θ is feasible, we deduce that $G(A)$ is a basic R^* -cyclic representation of A . This implies the "if" part of Theorem 8.2.

At last, let us analyse the running time of the procedure RCyclic. By Lemma 7.11, the computation of D requires time $O(nm\alpha)$. Let $k = |V_c \cup \overline{\text{Sink}(D')}|$, $n_0 = |R| + 2$ and $n_\ell = |E_\ell| + 2$ for all $E_\ell \in V_c \cup \overline{\text{Sink}(D')}$. Denote by α_0 (respectively, α_ℓ) the number of nonzero elements of the matrix $O(R^*)$ (respectively, N_ℓ) for all $1 \leq \ell \leq b$ such that $E_\ell \in V_c \cup \overline{\text{Sink}(D')}$. Since $\sum_{E_\ell \in V_c \cup \overline{\text{Sink}(D')}} n_\ell + n_0 = n + 2(k + 1)$ and $\sum_{E_\ell \in V_c \cup \overline{\text{Sink}(D')}} \alpha_\ell + \alpha_0 \leq 3\alpha$, using Theorem 2.5, step 2 in the subroutine GASink and step 3 in the procedure RCyclic altogether take time at most

$$\sum_{E_\ell \in V_c \cup \overline{\text{Sink}(D')}} n_\ell \alpha_\ell + n_0 \alpha_0 \leq (3n)(3\alpha) = Cn\alpha,$$

for some constant C . All other steps perform in time $O(\alpha)$. This concludes the proof of Theorem 8.1. ■

Chapter 9

Recognizing $\frac{1}{2}$ -binet matrices

Let A be a matrix of size $n \times m$ with 0, 1 or 2 coefficients and α the number of nonzero elements in A . In this chapter, we address the problem of recognizing whether A is $\frac{1}{2}$ -binet. More generally, given A and a row index subset Q of A , we describe a polynomial-time procedure, called `OnehalfbinetQ`, that computes a $\frac{1}{2}$ -binet representation of A such that each element in Q is a basic half-edge index, or determines that none exists. We will prove the following theorem.

Theorem 9.1 *Let Q be a row index subset of A . The matrix A can be tested for having a $\frac{1}{2}$ -binet representation such that each element in Q is a basic half-edge index, in time $O(nm^2\alpha)$, by the procedure `OnehalfbinetQ`.*

Let $S_2 = \{j : s_2(A_{\bullet j}) \neq \emptyset\}$. A family \mathcal{J} of row index subsets of A is said to be 1-forest if one can prove the following: If A is $\frac{1}{2}$ -binet, then in any $\frac{1}{2}$ -binet representation of A , each element of \mathcal{J} corresponds to the edge index set of a basic (negative) 1-tree and the elements of \mathcal{J} are pairwise disjoint.

Our first aim is to compute some 1-forest family \mathcal{J} such that if A is $\frac{1}{2}$ -binet, then there exists a $\frac{1}{2}$ -binet representation of A in which all nonbasic bidirected edges (except half-edges) correspond to columns of A whose support intersects two distinct elements of \mathcal{J} . Then, we reduce our problem to the recognition of an $\{i\}$ -cyclic matrix for some row index i .

9.1 An informal sketch of a recognition procedure

We first illustrate and comment the procedure `OnehalfbinetQ` with input the $\frac{1}{2}$ -binet matrix A given in Figure 9.1 and the empty row index subset ($Q = \emptyset$). We have to build up a $\frac{1}{2}$ -binet representation of A without knowing that such a representation exists. Since the first column of A has a nonempty 2-support, this implies the following. Provided that A has a binet representation $G(A)$, f_1 is a bidirected non-half 1-edge whose fundamental circuit contains a basic half-edge, and the set $R_1 = s(A_{\bullet 1}) = \{1, 2\}$ is the edge index set of a 1-tree. The procedure `OnehalfbinetQ` sets $\mathcal{J} = \{R_1\}$ which is 1-forest and $H = H(A) \setminus \{1\}$.

Then consider the submatrix $A_{\bullet V(H)}$. We notice that it is not a network matrix, since it contains the submatrix $A_{\{3,4,5\} \times \{3,4,5\}}$ of determinant 2. The second step consists in computing a minimal column index subset J such that $A_{\bullet J}$ is not a network matrix. For instance,

also removed from H , since the support of $A_{\bullet 9}$ intersects two elements of \mathcal{J} , namely Q_4 and Q_6 . Following this way, the procedure computes a graph $H \subseteq H(A)$ and a 1-forest family \mathcal{J} such that $A_{\bullet V(H)}$ is a network matrix decomposable into blocks, and the row index set of any block intersects at most one element in \mathcal{J} . This implies the following. There exists a $\frac{1}{2}$ -binet representation $G(A_{\bullet V(H)})$ of $A_{\bullet V(H)}$ such that all 2-edges in $G(A_{\bullet V(H)})$ are directed, and the basic half-edge of any 1-tree with edge index set in \mathcal{J} is entering (see Figure 9.3).

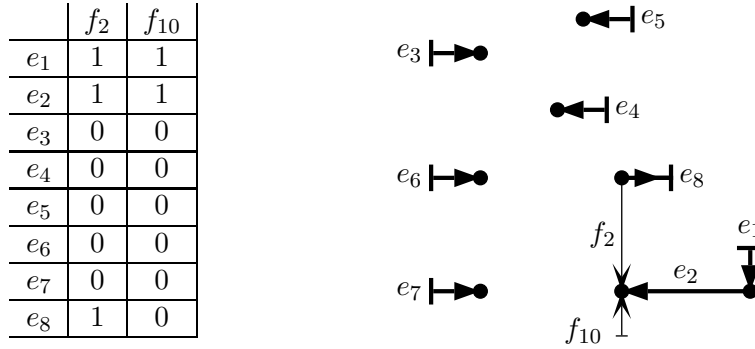


Figure 9.3: The binet matrix $A_{\bullet V(H)}$ and a $\frac{1}{2}$ -binet representation $G(A_{\bullet V(H)})$ of $A_{\bullet V(H)}$, where $V(H) = \{2, 10\}$, such that every 2-edge is directed and the basic half-edge of any 1-tree with edge index set in \mathcal{J} is entering, with $\mathcal{J} := \{R_1, Q_3, Q_4, Q_5, Q_6, Q_7\}$.

At last, let $S(\mathcal{J})$ be the index set of columns whose support intersects two elements of \mathcal{J} . Whenever A is binet, it will be proved that there exists a $\frac{1}{2}$ -binet representation $G(A)$ of A such that $S(\mathcal{J}) \cup S_2$ is the index set of nonbasic bidirected edges (except half-edges), and for each $R \in \mathcal{J}$, the basic half-edge with index in R is entering. Let δ be the row vector of size m given by $\delta_j = \begin{cases} 1 & \text{if } j \in S(\mathcal{J}) \cup S_2 \\ 0 & \text{Otherwise} \end{cases}$ and $A' = \begin{bmatrix} A \\ \delta \end{bmatrix}$.

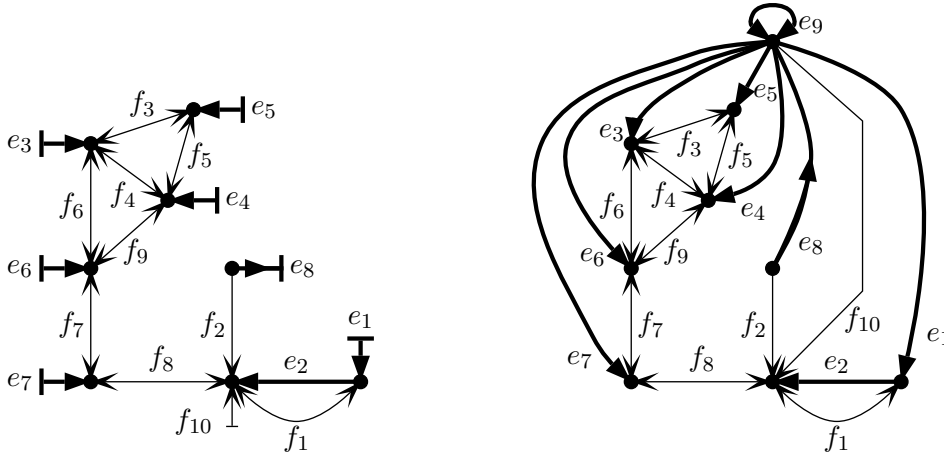


Figure 9.4: A $\frac{1}{2}$ -binet representation $G(A)$ of A and a $\{n+1\}$ -cyclic representation of A' .

One claims that A is $\frac{1}{2}$ -binet if and only if A' is $\{n+1\}$ -cyclic. Indeed, suppose that A is $\frac{1}{2}$ -binet and let $G(A)$ be a $\frac{1}{2}$ -binet representation of A as described above. Let us construct a $\{n+1\}$ -cyclic representation of A' as follows. Create a new vertex v and a loop entering v . Then, replace each half-edge entering (respectively, leaving) a node, say v' , by a directed edge $]v, v']$ (respectively, $]v', v]$). Conversely, given a $\{n+1\}$ -cyclic representation of the matrix A' , by contracting the basic loop, one obtains a $\frac{1}{2}$ -binet representation of A . See Figure 9.4.

9.2 The procedure OnehalfbinetQ

In this section, we deal with the general framework of the recognition problem. A column index subset J of A is said to be *odd-cyclic* if and only if the subgraph of $H(A)$ induced by J is an odd cycle of length at least 5, or a triangle such that $s(A_{\bullet j_1}) \cap s(A_{\bullet j_2}) \cap s(A_{\bullet j_3}) = \emptyset$ where $J = \{j_1, j_2, j_3\}$. Observe that in the graph $H(A)$, the vertex set of a triangle may be not odd-cyclic. Before describing a procedure for our recognition problem, we state an auxiliary lemma and the main theorem.

Lemma 9.2 *If A is a $\{0,1\}$ -matrix having a $\frac{1}{2}$ -binet representation with exactly one or two basic maximal 1-trees, then A is a network matrix.*

Proof. Let $G(A)$ be a $\frac{1}{2}$ -binet representation of A with exactly one or two basic maximal 1-trees. By switching operations if necessary, we may suppose that one basic half-edge of $G(A)$ is entering and the other one (if one exists) is leaving. Since any nonbasic nonhalf and bidirected 1-edge would have a non-empty 2-support, it follows that all edges except half-edges are directed (see Lemmas 4.1 and 4.2).

Then create a new vertex v and replace each half-edge entering (respectively, leaving) a node, say v' , by a directed edge from v to v' (respectively, v' to v). The obtained bidirected graph is a network representation of A . See Figure 9.5. ■

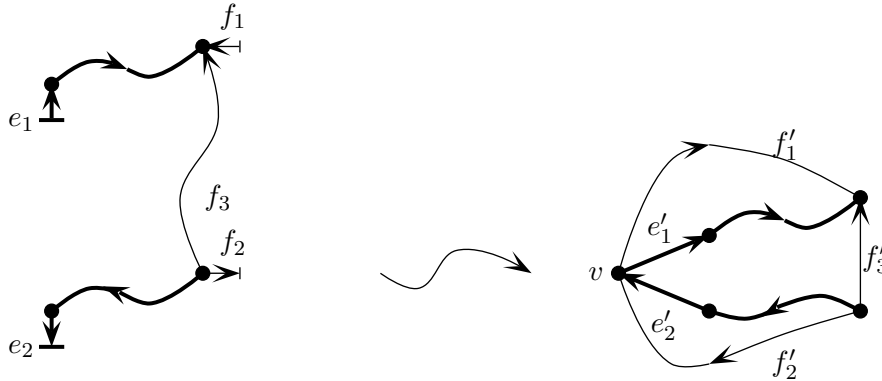


Figure 9.5: How to obtain a network representation of a matrix A , given a $\frac{1}{2}$ -binet representation of A in which all nonbasic edges except half-edges are directed. An edge e_i , respectively f_j , is replaced by the directed edge e'_i , respectively f'_j .

Theorem 9.3 *Suppose that A is a $\frac{1}{2}$ -binet $\{0, 1\}$ -matrix. Then A is a network matrix if and only if each column index subset of A is not odd-cyclic.*

Proof. Let $G(A)$ be a $\frac{1}{2}$ -binet representation of A .

Suppose that there exists an odd-cyclic set, say $\{1, \dots, d\}$, in $H(A)$. From the definition of an odd-cyclic set, it follows that $s(A_{\bullet j}) \cap s(A_{\bullet k}) \cap s(A_{\bullet l}) = \emptyset$ for any triplet $\{j, k, l\} \subseteq \{1, \dots, d\}$. By a relabeling of the edges we may suppose that e_k (corresponding to the k th row of A) is a basic edge contained in the fundamental circuit of f_k and f_{k+1} for $k = 1, \dots, d-1$, and e_d is in the fundamental circuit of f_d and f_1 . Then, since d is odd, the submatrix

$$A_{\{1, \dots, d\}^2} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

of A has a determinant equal to 2. Thus A is not totally unimodular, and so not a network matrix by Theorem 2.1.

Conversely, suppose that each column index subset of A is not odd-cyclic. We may assume that e_1, \dots, e_r are the basic half-edges of $G(A)$ and let $L_{G(A)} = (\{e_1, \dots, e_r\}, L)$ be the graph such that for $1 \leq i, i' \leq r$, $(e_i, e_{i'}) \in L$ if and only if the basic half-edges e_i and $e_{i'}$ in $G(A)$ are in the fundamental circuit of a 2-edge. Since an odd cycle in $L_{G(A)}$ would imply the existence of an odd-cyclic set, it results from the hypothesis that $L_{G(A)}$ is bipartite. Let $B_1 \uplus B_2$ be a bipartition of $\{e_1, \dots, e_r\}$ into two colour classes.

Using switching operations, one can construct a new $\frac{1}{2}$ -binet representation $G'(A)$ such that all half-edges belonging to B_1 (respectively, B_2) are entering (respectively, leaving). So, since the fundamental circuit of any 2-edge contains an entering and a leaving basic half-edge, all 2-edges are directed. We obtain from $G'(A)$ a network representation of A by creating a new vertex v and replacing each half-edge by a directed edge incident with v as illustrated in Figure 9.5. ■

The first step of the recognition procedure deals with the columns of A having an entry equal to 2. Let $S_2 = \{j : s_2(A_{\bullet j}) \neq \emptyset\}$. Denote by C_1, \dots, C_{l_0} the connected components of $H(A_{\bullet S_2})$ and we note $R_l := \bigcup_{j \in V(C_l)} s(A_{\bullet j})$ for all $1 \leq l \leq l_0$. In the next lemma, we give the graphical interpretation of these sets.

Lemma 9.4 *Suppose that the matrix A has a $\frac{1}{2}$ -binet representation $G(A)$. Then the family $\{R_1, \dots, R_{l_0}\}$ is 1-forest.*

Proof. It is immediate from the definition of $H(A_{\bullet S_2})$ that the sets R_1, \dots, R_{l_0} are pairwise disjoint. Further, by Corollary 3.6 and Lemma 4.1, for all $j \in S_2$, the nonbasic edge f_j is a 1-edge whose fundamental circuit contains a basic half-edge. So for all $1 \leq l \leq l_0$, since C_l is a connected subgraph of $H(A)$, R_l is the edge index set of a 1-tree. ■

Furthermore, we will search for odd-cyclic vertex sets in $H(A_{\bullet \overline{S_2}})$. The following lemma shows that if A is $\frac{1}{2}$ -binet, then nonbasic edges with index in an odd-cyclic subset of $\overline{S_2}$ are 2-edges.

Lemma 9.5 *Suppose that A has a $\frac{1}{2}$ -binet representation $G(A)$. Then, in any $\frac{1}{2}$ -binet representation of A , each odd-cyclic subset of $\overline{S_2}$ is an index set of 2-edges.*

Proof. Let S be an odd-cyclic subset of $\overline{S_2}$. We may assume that $S = \{1, \dots, d\}$. By Theorem 9.3, we deduce that the matrix $A_{\bullet S}$ is not a network matrix. (*)

Up to a relabeling of the edges, we may assume that the subgraph of $H(A)$ induced by $\{1, \dots, d\}$ is the cycle $(1, (1, 2), 2, \dots, d, (d, 1), 1)$. If all f_j ($1 \leq j \leq d$) are 1-edges, then since $(j, j+1) \in H(A)$ for $j = 1, \dots, d-1$, the basic subgraph of $G(A)$ with edge index set $\cup_{j=1}^d s(A_{\bullet j})$ is connected. If we delete all nonbasic edges with index in \overline{S} , we obtain a $\frac{1}{2}$ -binet representation of $A_{\bullet S}$ with exactly one basic maximal 1-tree, so using Lemma 9.2 that contradicts (*).

Denote by $f_{j_1}, f_{j_2}, \dots, f_{j_t}$ the succession of 2-edges with index between 1 and d so that $1 \leq j_1 < j_2 < \dots < j_t \leq d$. If $t = 1$, then using Lemma 9.2 it contradicts the observation (*). Assume $t = 2$. If f_{j_1} and f_{j_2} have two common basic half-edges in their fundamental circuits, using Lemma 9.2 we also obtain a contradiction. Otherwise, let e_{h_1} and e_{h_2} (respectively, e_{h_2} and e_{h_3}) be the basic half-edges in the fundamental circuit of f_{j_1} (respectively, f_{j_2}). Denote by T_1 , T_2 and T_3 the basic maximal 1-trees containing e_{h_1} , e_{h_2} and e_{h_3} , respectively. Then, any 1-edge f_j with $1 \leq j < j_1$ (respectively, $j_t < j \leq d$) has its endnodes in T_1 (respectively, T_3). Hence 1 is not adjacent to d in $H(A)$, except if $1 = j_1$ and $d = j_2$. So $1 = j_1$ and $d = j_2$. Since all edges f_j with $1 < j < d$ are 1-edges with endnodes in T_2 , one can construct a network representation of $A_{\bullet S}$, using the same transformation as illustrated in Figure 9.5. This is in contradiction with (*). Thus $t \geq 3$.

Since there is a path from j_1 to j_2 in $H(A)$, all of whose inner vertices are indexes of 1-edges, it follows that f_{j_1} and f_{j_2} have a common basic half-edge in their fundamental circuits. So $(j_1, j_2) \in H(A)$ and in a same way, one can prove that $(j_1, (j_1, j_2), j_2, \dots, j_t, (j_t, j_1), j_1)$ is a cycle in $H(A)$. Since the cycle $(1, (1, 2), 2, \dots, d, (d, 1), 1)$ has no chord in $H(A)$, we deduce that $l = d$, which concludes the proof. ■

In what follows, if $\{j_1, \dots, j_t\}$ is an odd-cyclic set, then we will assume that the subgraph of $H(A)$ induced by this set is equal to $(j_1, (j_1, j_2), j_2, \dots, j_t, (j_t, j_1), j_1)$. Denote by $Q_{j_l} = s(A_{\bullet j_l}) \cap s(A_{\bullet j_{l+1}})$ for $l = 1, \dots, t-1$ and $Q_{j_t} = s(A_{\bullet j_t}) \cap s(A_{\bullet j_1})$.

Lemma 9.6 *Suppose that the matrix A has a $\frac{1}{2}$ -binet representation $G(A)$. Let $\{j_1, \dots, j_t\} \subseteq \overline{S_2}$ be an odd-cyclic set. Then the family $\{Q_{j_1}, \dots, Q_{j_t}\}$ is 1-forest.*

Proof. From the definition of an odd-cyclic set, it results that the sets Q_{j_1}, \dots, Q_{j_t} are pairwise disjoint and $t \geq 3$. By Lemma 9.5, for all $l = 1, \dots, t$, f_{j_l} is a 2-edge. So, if the fundamental circuits of f_{j_1} and f_{j_2} for instance share two common basic half-edges, then $s(A_{\bullet j_1}) \cap s(A_{\bullet j_2}) \cap s(A_{\bullet j_3}) \neq \emptyset$ which contradicts the definition of an odd-cyclic set. Thus the fundamental circuits of f_{j_1} and f_{j_2} have exactly one common basic half-edge. Therefore Q_{j_1} is the edge index set of a negative 1-tree. Similarly, Q_{j_l} is the edge index set of a negative 1-tree, for $l = 2, \dots, t$. ■

In the following procedure, since it is hard to find an odd cycle of length at least 5 in a graph or prove that none exists, odd-cyclic sets are discovered by searching for minimal column subsets of A forming a non-network matrix. Odd-cyclic vertex sets are removed out

of a subgraph H of $H(A)$, until H has no odd-cyclic subset any more. On the other hand, any row index subset Q_{j_l} (for some column index j_l) which does not intersect any element of \mathcal{J} is added in \mathcal{J} .

Procedure S2OddCycle(A)

Input: A matrix A .

Output: Either a graph H and a family \mathcal{J} updated, or determines that A is not $\frac{1}{2}$ -binet.

- 1) let $\mathcal{J} = \{R_1, \dots, R_{l_0}\}$ and $H = H(A) \setminus S_2$;
- 2) **while** $A_{\bullet V(H)}$ is not a network matrix, **do**
- 3) search for a minimal subset $J = \{j_1, \dots, j_t\}$ in $V(H)$ such that $A_{\bullet J}$ is not a network matrix;
- 4) if J is not odd-cyclic, then STOP: output that A is not $\frac{1}{2}$ -binet;
- 5) for $l = 1, \dots, t$, if $Q_{j_l} \cap R = \emptyset$ for all $R \in \mathcal{J}$, then add Q_{j_l} in \mathcal{J} ;
- 6) $H = H \setminus \{j_1, \dots, j_t\}$;
- endwhile**
- output H and \mathcal{J} ;

Suppose that A is $\frac{1}{2}$ -binet and the procedure S2OddCycle has output a graph H and a family \mathcal{J} . One idea is to prove the existence of a $\frac{1}{2}$ -binet representation of A , if one exists, such that all bidirected 2-edges are known precisely. By assuming that the matrix $A_{\bullet V(H)}$ is decomposed into (connected) blocks, provided that there exist two elements of \mathcal{J} that are row index subsets of a same block, it seems inconvenient to find a $\frac{1}{2}$ -binet representation in which all bidirected 2-edges are known. For avoiding this case, we will consider a procedure called Path.

For a given \mathcal{J} and a subgraph H of $H(A)$, a vertex j of H is *marked* if and only if there exists at least one element R of \mathcal{J} such that $s(A_{\bullet j}) \cap R \neq \emptyset$. A path from a vertex j to j' in H is called *linking* if all its nodes are not marked except j and j' . Moreover, if $j = j'$, then one requires the existence of two sets $R, R' \in \mathcal{J}$ ($R \neq R'$) such that $s(A_{\bullet j}) \cap R \neq \emptyset$ and $s(A_{\bullet j}) \cap R' \neq \emptyset$, otherwise for any $R, R' \in \mathcal{J}$ such that $s(A_{\bullet j}) \cap R \neq \emptyset$ and $s(A_{\bullet j'}) \cap R' \neq \emptyset$, we have $s(A_{\bullet j'}) \cap R = \emptyset$ and $s(A_{\bullet j}) \cap R' = \emptyset$. Let us see a useful lemma.

Lemma 9.7 *Suppose that A has a $\frac{1}{2}$ -binet representation $G(A)$. Let \mathcal{J} be a 1-forest family of row index subsets of A and H a subgraph of $H(A)$. Then, all vertices of a minimal linking path in H correspond to indexes of 2-edges in $G(A)$.*

Proof. Let $\gamma = (j_1, (j_1, j_2), j_2, \dots, j_{t-1}, (j_{t-1}, j_t), j_t)$ be a minimal linking path in H . If $t = 1$, the conclusion is clear. Now assume $t \geq 2$. Let $R, R' \in \mathcal{J}$ ($R \neq R'$) such that $s(A_{\bullet j_1}) \cap R \neq \emptyset$ and $s(A_{\bullet j_t}) \cap R' \neq \emptyset$. Since \mathcal{J} is a 1-forest family, we may denote by T and T' the basic 1-trees in $G(A)$ with edges index sets R and R' , respectively. Denote by e_i and $e_{i'}$ the half-edges of T and T' , respectively. Since $T \neq T'$ and using the definition of H , e_i as well as $e_{i'}$ have to be contained in the fundamental circuit of a 2-edge with index in γ . By definition of a minimal linking path, we deduce that f_{j_1} is the unique nonbasic edge whose fundamental circuit contains e_i , so f_{j_1} is a 2-edge. For a same reason, f_{j_t} is a 2-edge.

Now, suppose that there exists a vertex in γ corresponding to the index of a 1-edge in $G(A)$. We may assume that $f_{j_l}, f_{j_{l+1}}, \dots, f_{j_{l'}}$ are 1-edges and $f_{j_{l-1}}$ and $f_{j_{l'+1}}$ are 2-edges for

some $1 < l \leq l' < t$. It results that j_{l-1} and $j_{l'+1}$ are adjacent in H , which contradicts the minimality of γ . \blacksquare

For a path $(j_1, (j_1, j_2), j_2, \dots, j_{t-1}, (j_{t-1}, j_t), j_t)$ in $H(A)$, we denote by $Q_{j_l} = s(A_{\bullet j_l}) \cap s(A_{\bullet j_{l+1}})$ for $l = 1, \dots, t-1$. The procedure Path is stated below.

Procedure Path(A, H, \mathcal{J})

Input: A matrix A , a graph $H \subseteq H(A)$ and a family \mathcal{J} of row index subsets of A .

Output: A graph H and a family \mathcal{J} updated.

- 1) **while** there exists a minimal linking path in H , **do**
- 2) let $(j_1, (j_1, j_2), j_2, \dots, j_{t-1}, (j_{t-1}, j_t), j_t)$ be a minimal linking path in H ;
- 3) for $l = 1, \dots, t-1$, if $Q_{j_l} \cap R = \emptyset$ for all $R \in \mathcal{J}$, then add Q_{j_l} in \mathcal{J} ;
 $H = H \setminus \{j_1, \dots, j_t\}$;
- endwhile**
- output H and \mathcal{J} ;

Finally, for a given family \mathcal{J} of row index subsets of A , let $S(\mathcal{J})$ be the set of vertices j in $H(A)$ such that $s(A_{\bullet j}) \cap R \neq \emptyset$ and $s(A_{\bullet j}) \cap R' \neq \emptyset$ for some $R, R' \in \mathcal{J}$ ($R \neq R'$). We define the matrix $A' = \begin{bmatrix} A \\ \delta \end{bmatrix}$, where δ is a row vector given by $\delta_j = \begin{cases} 1 & \text{if } j \in S(\mathcal{J}) \cup S_2 \\ 0 & \text{Otherwise} \end{cases}$. Now one can describe a main procedure and prove its correctness.

Procedure Onehalfbinet(A)

Input: A matrix A with 0, 1, or 2 entries.

Output: Either a $\frac{1}{2}$ -binet representation $G(A)$ of A , or determines that none exists.

- 1) call **S2OddCycle**(A) outputting some $H \subseteq H(A)$ and \mathcal{J} , or the fact that A is not $\frac{1}{2}$ -binet;
- 2) call **Path**(A, H, \mathcal{J}) outputting some $H \subseteq H(A)$ and \mathcal{J} ;
- 3) construct the matrix A' as above and call **RCyclic**($A', \{n+1\}$) of Section 8.2;
if we have a $\{n+1\}$ -cyclic representation $G(A')$, then go to 4,
otherwise **STOP**: output that A is not $\frac{1}{2}$ -binet;
- 4) output the bidirected graph $G(A)$ obtained from $G(A')$ by contracting the basic loop;

Theorem 9.8 *The output of the procedure Onehalfbinet is correct.*

Proof. Suppose that A has a $\frac{1}{2}$ -binet representation $G(A)$. Using Theorem 9.3, we deduce that the subroutine **S2OddCycle** does not stop in its step 4. Let \mathcal{J} and H be obtained by performing successively the procedures **S2OddCycle** and **Path**. By Lemmas 9.4, 9.6 and 9.7, it results that \mathcal{J} is 1-forest. Thus, for any $R \in \mathcal{J}$, R is the edge index set of a basic 1-tree in $G(A)$. Moreover, by construction, we have the equality

$$\{1, \dots, m\} = S_2 \uplus S(\mathcal{J}) \uplus V(H).$$

By the subroutine **S2OddCycle**, the matrix $A_{\bullet V(H)}$ is a network matrix. Moreover, thanks to the subroutine **Path**, for $R, R' \in \mathcal{J}$ ($R \neq R'$), the basic 1-trees with edge index sets R and R' are in different connected components of the subgraph $G(A_{\bullet V(H)})$. So, up to switching

at all nodes of some basic maximal 1-trees, we may assume that for all $R \in \mathcal{J}$, the basic half-edge with index in R is entering and all 2-edges in $G(A_{\bullet V(H)})$ are directed (see also the proof of Theorem 9.3). Thus, up to switching operations, we may assume that $G(A)$ is a $\frac{1}{2}$ -binet representation such that $S(\mathcal{J})$ corresponds to the index set of bidirected 2-edges and each basic half-edge with index in some $R \in \mathcal{J}$ is entering.

Using $G(A)$, we construct a $\{n+1\}$ -cyclic representation of A' as follows. Create a new vertex v and a loop entering v corresponding to the last row of A' . Then replace each half-edge which is entering (respectively, leaving) a node say v' , by a directed edge from v to v' (respectively, v' to v). By Theorem 8.1, the procedure RCyclic computes a $\{n+1\}$ -cyclic representation of A' in step 3.

Whatever input, it is straightforward that the output of the procedure Onehalfbinet (if it does not stop) is a $\frac{1}{2}$ -binet representation of A . \blacksquare

For any positive integer q , we define

$$N^{(1)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N^{(q)} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \\ 0 & & \cdots & & 0 \\ \vdots & & & & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix},$$

where $N^{(1)}$ is of size $(n+2) \times 3$, and $N^{(q)}$ of size $(n+1) \times (q+1)$ (respectively, $n \times q$) if q is even (respectively, odd and at least equal to 3). Then, we define

$$A^{(1)} = \begin{pmatrix} 0_{2 \times m} & N^{(1)} \\ A & \end{pmatrix} \quad \text{and} \quad A^{(q)} = \begin{pmatrix} 0_{(q+1) \bmod 2 \times m} & N^{(q)} \\ A & \end{pmatrix} \text{ for } q \geq 2.$$

The relation between A and $A^{(q)}$ appears in the following lemma.

Lemma 9.9 *The matrix A has a $\frac{1}{2}$ -binet representation such that the first q rows correspond to (basic) half-edges if and only if $A^{(q)}$ has a $\frac{1}{2}$ -binet representation.*

Proof. The "only if" part is not difficult and the "if" part follows from Lemma 9.5. \blacksquare

From Lemma 9.9, we easily deduce the following procedure that recognizes whether the matrix A has a $\frac{1}{2}$ -binet representation such that each element in Q is a basic half-edge index, where Q is a given row index subset of A .

Procedure OnehalfbinetQ(A, Q)

Input: A matrix A with entries 0, 1, or 2 and a row index subset Q .

Output: A $\frac{1}{2}$ -binet representation $G(A)$ such that each element in Q corresponds to a basic half-edge index, or determines that none exists.

1) let $q = |Q|$, $\tilde{A} = A$ and permute the rows of \tilde{A} so that those with index in Q

- appear first;
- 2) call `Onehalfbinet`($\tilde{A}^{(q)}$);
 if the procedure `Onehalfbinet` outputs a $\frac{1}{2}$ -binet representation $G(\tilde{A}^{(q)})$, then let $G(\tilde{A})$ be the graph obtained from $G(\tilde{A}^{(q)})$ by deleting the non-basic edges corresponding to the columns of $N^{(q)}$ and basic ones corresponding to rows with n first zero entries, otherwise STOP: output that A has no $\frac{1}{2}$ -binet representation in which each element of Q corresponds to a basic half-edge index;
 up to a relabeling of basic edges, output a $\frac{1}{2}$ -binet representation $G(A)$ of A ;

Finally, we can prove Theorem 9.1.

Proof of Theorem 9.1. The correctness of the procedure `OnehalfbinetQ` follows from Lemma 9.9 and Theorem 9.8. We know by Theorem 2.5 that determining whether a given matrix of size $n \times m$ with β nonzero entries is a network matrix takes time $O(n\beta)$. Thus step 3 in the subroutine `S2OddCycle` works in time $O(nm\alpha)$. Then, the number of passages through step 2 in the same subroutine does not exceed m . Hence the subroutine `S2OddCycle` takes time $O(nm^2\alpha)$. It is not difficult to see that the subroutine `Path` takes time at most $O(nm^2)$, and by Theorem 8.1, the procedure `RCyclic` works in time $O(nm\alpha)$. This concludes the proof. ■

Chapter 10

Recognizing bicyclic matrices

In this Chapter, we provide a characterization of nonnegative $\frac{1}{2}$ -equisupported bicyclic matrices as well as a recognition procedure called *Bicyclic* for these matrices. Let A be a connected $\frac{1}{2}$ -equisupported matrix of size $n \times m$ with entries 0, 1, $\frac{1}{2}$ or 2 and at least one $\frac{1}{2}$ -entry. Let α be the number of nonzero elements in A . A proof of the following theorem will be given.

Theorem 10.1 *The matrix A can be tested for being bicyclic by the procedure *Bicyclic*. The computational effort required is $O(nm\alpha)$.*

If A has a bicyclic representation $G(A)$, then since A is $\frac{1}{2}$ -equisupported, using Corollary 3.6 and Lemma 4.1 it follows that every column with a nonempty $\frac{1}{2}$ -support corresponds to a 2-edge; and the $\frac{1}{2}$ -support of a 2-edge is equal to the edge index set of both basic cycles in $G(A)$. Furthermore, each basic cycle corresponds to a (closed) consistently oriented path.

Let $S_{\frac{1}{2}} = \{j : s_{\frac{1}{2}}(A_{\bullet j}) \neq \emptyset\}$, $R^* = s_{\frac{1}{2}}(A_{\bullet j})$ for any $j \in S_{\frac{1}{2}}$, and $S^* = \{j : s(A_{\bullet j}) \cap R^* \neq \emptyset\}$. Let us partition all row indexes of A into subsets C_1, \dots, C_r called *cells* so that i and i' belong to a same cell if and only if $i \in s(A_{\bullet j})$, $i' \in s(A_{\bullet j'})$, and j and j' are in a same connected component of $H(A) \setminus S_{\frac{1}{2}}$. For $k = 1, \dots, r$, let

$$R_k = C_k \cap R^* \quad \text{and} \quad A_k = A_{C_k \times f(C_k)}.$$

The set R_k ($1 \leq k \leq r$) is called an *interval*. Up to a renumbering of the cells, we may note R_1, \dots, R_{ξ} the nonempty intervals. Let

$$\mathcal{K} = \{C_1, \dots, C_{\xi}\} \quad \text{and} \quad F = \{j : s(A_{\bullet j}) \subseteq \cup_{k=\xi+1}^r C_k\}.$$

If $A_k^{\frac{1}{2} \rightarrow 1}$ is not a network matrix or $R_k \neq \cup_{j \in S_{\frac{1}{2}}} s(A_{\bullet j}) \cap C_k$ for some $1 \leq k \leq \xi$, then the cell C_k is called *central*. For a matrix N and a row index subset R of N , an R $\frac{1}{2}$ -binet representation of N is a $\frac{1}{2}$ -binet representation of N such that R is the index set of all basic half-edges. A matrix is called R $\frac{1}{2}$ -binet if and only if it has an R $\frac{1}{2}$ -binet representation. Later, we shall study some bipartitions of \mathcal{K} , denoted by $\Sigma(\mathcal{K})$, and construct submatrices $M_I(\Sigma)$ and $M_{II}(\Sigma)$ of A and a matrix $N(\Sigma)$ with respect to $\Sigma(\mathcal{K})$. For $i = I$ and II , $R_i^*(\Sigma)$ will denote the row indexes of $M_i(\Sigma)$ belonging to R^* . We shall denote by n_I and n_{II} two particular row indexes of $N(\Sigma)$. Then a proof of the following theorem will be given.

Theorem 10.2 *The matrix A is bicyclic if and only if there exists a bicompatible bipartition $\Sigma(\mathcal{K})$ such that for any bipartition $\Sigma'(\mathcal{K})$ equivalent to $\Sigma(\mathcal{K})$, the matrix $M_i(\Sigma')$ is $E_i(\Sigma')$ -cyclic for $i = I$ and II and $N(\Sigma')$ is $\{n_I, n_{II}\} \frac{1}{2}$ -binet.*

In Section 10.1, we provide some intuitions and ideas about Theorems 10.1 and 10.2 using an example. Then a formal proof of these theorems is given in Section 10.2.

10.1 An informal sketch of a recognition procedure

Before embarking on the proof of Theorems 10.1 and 10.2, we describe the graphical idea on which these are based using the example of Figure 10.1. As explained above, we partition the row indexes of A into *cells* C_1, \dots, C_r in the following way: two row indexes i and i' belong to a same cell if and only if $i \in s(A_{\bullet j})$, $i' \in s(A_{\bullet j'})$, and j and j' are in a same connected component of $H(A) \setminus S_{\frac{1}{2}}$. Up to a renumbering of cells, we have $C_1 = \{1, 3, 4\}$, $C_2 = \{6, 7\}$, $C_3 = \{2, 5\}$, $C_4 = \{8, 9\}$, $C_5 = \{10\}$ and $C_6 = \{11\}$. Provided that A has a bicyclic representation $G(A)$, each cell C_k ($1 \leq k \leq r$) corresponds to the edge index set of a (basic) tree or 1-tree in $G(A)$ which is called a *cell* and denoted as $T(C_k)$ (see Figure 10.2); moreover, a cell in $G(A)$ is called *central* if and only if its corresponding edge index set is central.

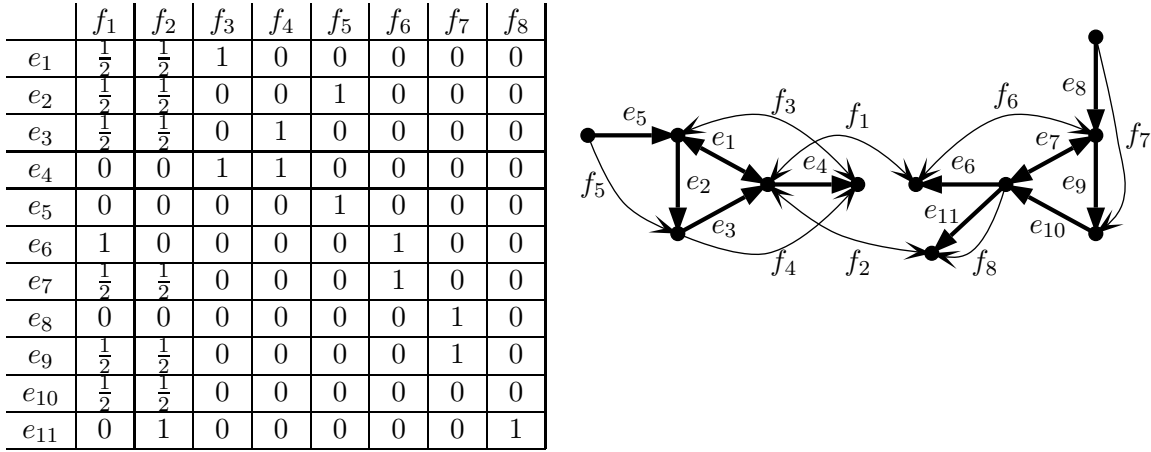


Figure 10.1: A bicyclic matrix A such that $R^* = \{1, 2, 3, 7, 9, 10\}$, $S_{\frac{1}{2}} = \{1, 2\}$, and a binet representation $G(A)$ of A .

Among all cells given in Figure 10.2, what are the central ones? We have $R_1 = C_1 \cap R^* = \{1, 3\}$, $R_2 = C_2 \cap R^* = \{7\}$ and

$$A_1^{\frac{1}{2} \rightarrow 1} = \begin{array}{c|cccc} & f_1 & f_2 & f_3 & f_4 \\ \hline e_1 & 1 & 1 & 1 & 0 \\ e_3 & 1 & 1 & 0 & 1 \\ e_4 & 0 & 0 & 1 & 1 \end{array}, \quad A_2^{\frac{1}{2} \rightarrow 1} = \begin{array}{c|ccc} & f_1 & f_2 & f_6 \\ \hline e_6 & 1 & 0 & 1 \\ e_7 & 1 & 1 & 1 \end{array}$$

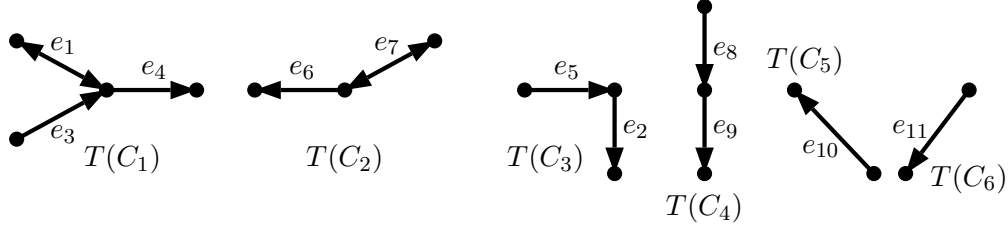


Figure 10.2: The different cells in the bicyclic representation $G(A)$ of the matrix A given in Figure 10.1.

The matrix $A_1^{\frac{1}{2} \rightarrow 1}$ has a submatrix of determinant 2, so it is not a network matrix. On the contrary, $A_2^{\frac{1}{2} \rightarrow 1}$ is a network matrix, but $R_2 = \{7\} \neq \{6, 7\} = \cup_{j \in S_{\frac{1}{2}}} s(A_{\bullet j}) \cap C_2$. Hence the cells C_1 and C_2 are central, and one can verify that the other ones are not. One can prove that this implies the following. Provided that A has a bicyclic representation $G(A)$, $T(C_1)$ and $T(C_2)$ contain each one at least one central edge. Indeed, $e_1, e_3 \in T(C_1)$ and $e_2 \in T(C_2)$ (see Figures 10.1 and 10.2). Nevertheless, a non-central cell in $G(A)$ might contain central edges, as evidenced by $T(C_5)$.

Once the procedure Bicyclic has located all central cells, the goal is to split up all cells into two groups, say \mathcal{K}_I and \mathcal{K}_{II} , satisfying the following condition. There exists a bicyclic representation $G(A)$ of A in which T_I and T_{II} are the basic 1-trees, and any cell in \mathcal{K}_i corresponds to the edge index set of a subgraph of T_i for $i = I$ and II . One difficulty is that the number of partitions of \mathcal{K} may be huge. However, we observe in Figure 10.1, that it is possible to "move" a non-central cell, for instance $T(C_5)$, from one basic maximal 1-tree to the other as illustrated in Figure 10.3, in order to obtain a new bicyclic representation of A . This motivates a notion of "equivalent" bipartitions for reducing the number of bipartitions that have to be considered.

Given a bipartition $\Sigma(\mathcal{K}) = \{\mathcal{K}_I, \mathcal{K}_{II}\}$ of \mathcal{K} with $\mathcal{K}_I = \{C_1, C_3\}$ and $\mathcal{K}_{II} = \{C_2, C_4, C_5\}$ for instance, we define $E_i(\Sigma) = \cup_{C_k \in \mathcal{K}_i} C_k$, $R_i^*(\Sigma) = E_i(\Sigma) \cap R^*$ and $M_i(\Sigma) = A_{E_i(\Sigma) f(E_i(\Sigma))}$, for $i = I$ and II , as well as a matrix $N(\Sigma)$ having two particular row indexes denoted by n_I and n_{II} . See Figure 10.4. Then one computes if possible an $E_i(\Sigma)$ -cyclic representation of $M_i(\Sigma)$, for $i = I$ and II , and a $\frac{1}{2}$ -binet representation of $N(\Sigma)$ whose basic half-edges are e_{n_I} and $e_{n_{II}}$. Provided that all these representations have been found, one can construct a bicyclic representation of A (see Figure 10.5).

10.2 The procedure Bicyclic

In this section, we describe the procedure Bicyclic and provide a proof of Theorems 10.1 and 10.2. Suppose that A has a bicyclic representation $G(A)$. For all $1 \leq k \leq \xi$, the interval R_k represents the edge index set of a basic loop, or a consistently oriented path, denoted as \mathcal{P}_k , in a basic cycle. For every column index j , the nonbasic edge f_j is a 2-edge if and only if $j \in S_{\frac{1}{2}}$. Observe that for all $1 \leq k \leq r$, the cell C_k is the edge index set of a (basic) tree or 1-tree of $G(A)$, denoted as $T(C_k)$, which is called a *cell*. We denote by T_I and T_{II} the

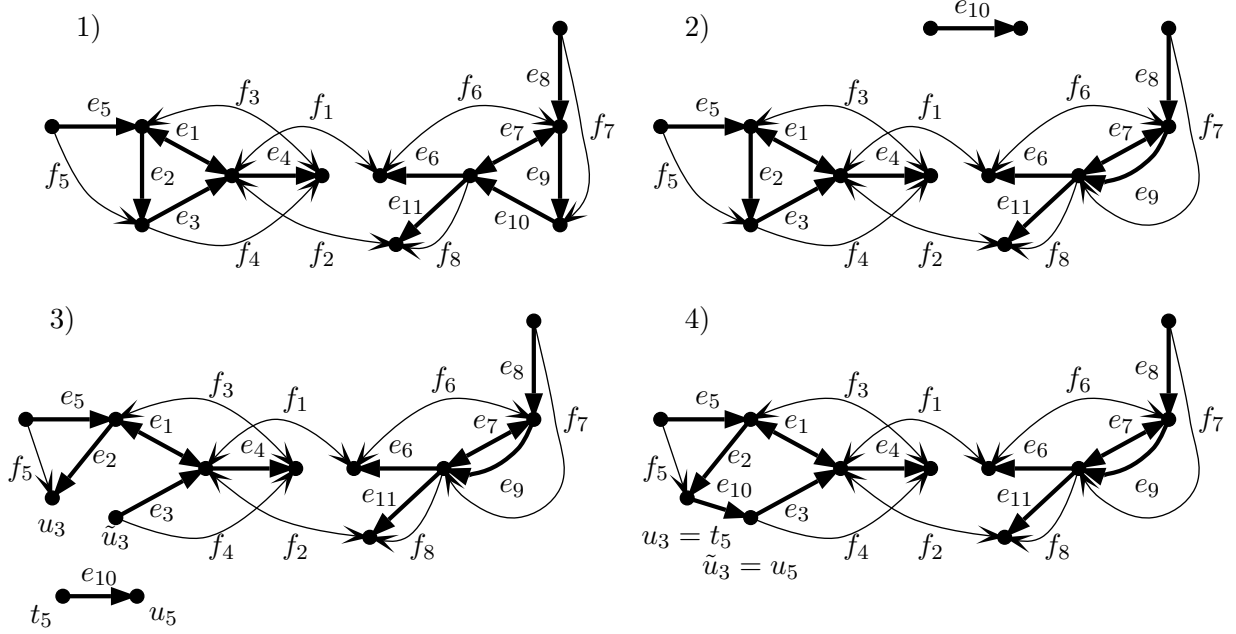


Figure 10.3: How to move a non-central cell in a bicyclic representation $G(A)$ of A from one basic 1-tree to the other, where A and $G(A)$ are given in Figure 10.1 (see pictures from 1 to 4). The vertices u_3 , \tilde{u}_3 , u_5 and t_5 are defined in the proof of Proposition 10.5.

$M_I(\Sigma) =$		f_1^I	f_2^I	f_3	f_4	f_5
	e_1	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0
	e_2	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1
	e_3	$\frac{1}{2}$	$\frac{1}{2}$	0	1	0
	e_4	0	0	1	1	0
	e_5	0	0	0	0	1

$M_{II}(\Sigma) =$		f_1^{II}	f_2^{II}	f_6	f_7
	e_6	1	0	1	0
	e_7	$\frac{1}{2}$	$\frac{1}{2}$	1	0
	e_8	0	0	0	1
	e_9	$\frac{1}{2}$	$\frac{1}{2}$	0	1
	e_{10}	$\frac{1}{2}$	$\frac{1}{2}$	0	0

$N(\Sigma) =$		f_1^I	f_2^I	f_1^{II}	f_2^{II}	f_1	f_2	f_8
	e_6	0	0	1	0	1	0	0
	e_{11}	0	0	0	0	0	1	1
	e_{n_I}	1	1	0	0	1	1	0
	$e_{n_{II}}$	0	0	1	1	1	1	0

Figure 10.4: An example of the matrices $M_I(\Sigma)$, $M_{II}(\Sigma)$ and $N(\Sigma)$ where A is given in Figure 10.1, $\Sigma(\mathcal{K}) = \{\mathcal{K}_I, \mathcal{K}_{II}\}$ with $\mathcal{K}_I = \{C_1, C_3\}$ and $\mathcal{K}_{II} = \{C_2, C_4, C_5\}$.

maximal basic 1-trees of $G(A)$. We say that $G(A)$ induces the bipartition $\Sigma(\mathcal{K}) = \{\mathcal{K}_I, \mathcal{K}_{II}\}$, where $\mathcal{K}_i = \{C_k : T(C_k) \subseteq T_i\}$ for $i = I$ and II . A cell in $G(A)$ is said to be *central* if and only if its edge index set is central. The following lemma gives an enlightenment on the notion of central cell.

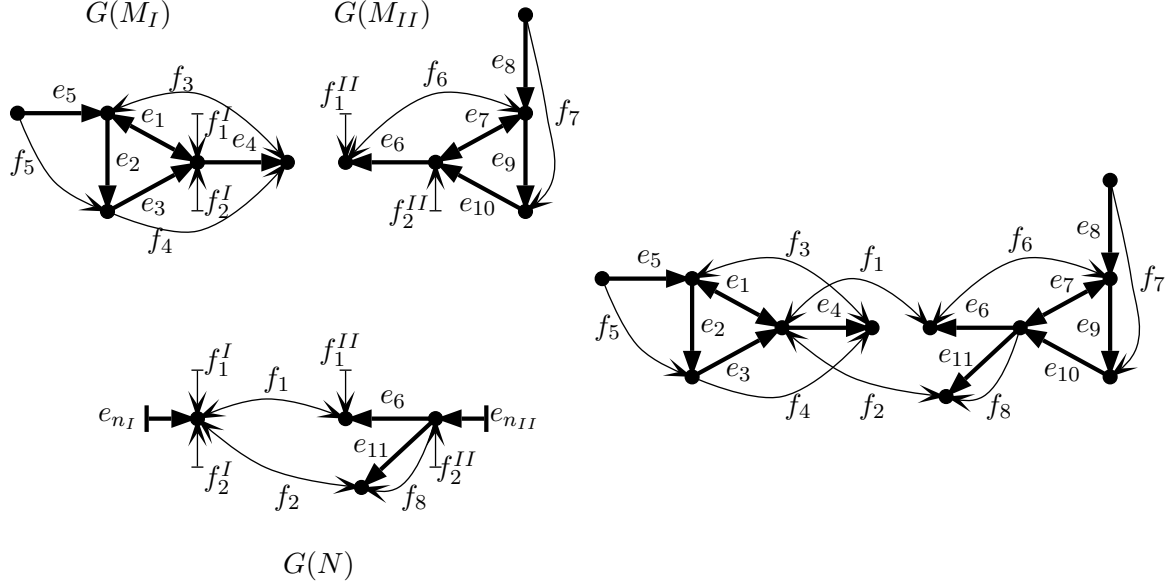


Figure 10.5: An $E_i(\Sigma)$ -cyclic representation of $M_i(\Sigma)$ for $i = I$ and II , a $\{n_I, n_{II}\}$ $\frac{1}{2}$ -binet representation of $N(\Sigma)$ and a bicyclic representation of A , where A is given in Figure 10.1 and $M_I(\Sigma)$, $M_{II}(\Sigma)$ and $N(\Sigma)$ in Figure 10.4.

Lemma 10.3 *Suppose that A has a bicyclic representation $G(A)$. Then every central cell in $G(A)$ contains a central edge.*

Proof. Let $T(C_k)$ be a central cell, for some $1 \leq k \leq \xi$, and suppose that it is contained in T_I . Since $C_k \cap R^* \neq \emptyset$, we may assume that the basic cycle of T_I is not a loop. Let us denote by e_1 and e_ρ the central edges of T_I so that e_1 is bidirected. We may also assume that $e_1 \notin \mathcal{P}_k$. Hence, $T(C_k)$ is a directed tree, and no column in A_k has a 2-entry, because the fundamental circuit of a column with a nonempty 2-support contains a whole basic cycle. Moreover, the intersection of the fundamental circuit of any 2-edge with $T(C_k)$ is a consistently oriented path. Therefore, $T(C_k)$ is a basic network representation of $A_k^{\frac{1}{2} \rightarrow 1}$.

From the definition of a central cell, it follows that $\cup_{j \in S_{\frac{1}{2}}} s(A_{\bullet j}) \cap C_k \neq R_k$. So there exists a basic edge e_i and a 2-edge f_j ($j \in S_{\frac{1}{2}}$), such that e_i is in the fundamental circuit of f_j and $i \in C_k \setminus R_k$. Then, since A is nonnegative, by Lemma 4.2 the stem issued from f_j in T_I contains the central node of T_I and e_i (see Figure 4.4). As $T(C_k)$ is connected and contains at least one edge of the basic cycle in T_I , we deduce that e_ρ belongs to $T(C_k)$. ■

A bipartition of \mathcal{K} into two subsets $\Sigma(\mathcal{K}) = \{\mathcal{K}_I, \mathcal{K}_{II}\}$ (where $\mathcal{K} = \mathcal{K}_I \uplus \mathcal{K}_{II}$) is said to be *bicompatible* if $\mathcal{K}_i \neq \emptyset$ and \mathcal{K}_i contains at most two central cells for $i = I$ and II . A bipartition $\Sigma'(\mathcal{K}) = \{\mathcal{K}'_I, \mathcal{K}'_{II}\}$ is called *equivalent* to a bipartition $\Sigma(\mathcal{K}) = \{\mathcal{K}_I, \mathcal{K}_{II}\}$, if $\Sigma'(\mathcal{K}) = \Sigma(\mathcal{K})$ or " \mathcal{K}_I and \mathcal{K}_{II} have each one at least one non-central cell, and one can obtain Σ' from Σ by moving some non-central cells between \mathcal{K}_I and \mathcal{K}_{II} and keeping at least one non-central cell in each subset". Clearly, this is an equivalence relation. Denote by \mathcal{S} the quotient set of all bicompatible bipartitions by this equivalence relation. Given a bipartition $\Sigma(\mathcal{K}) = \{\mathcal{K}_I, \mathcal{K}_{II}\}$,

we define

$$E_i(\Sigma) = \cup_{C_k \in \mathcal{K}_i} C_k, \quad R_i^*(\Sigma) = E_i(\Sigma) \cap R^*$$

and

$$M_i(\Sigma) = A_{E_i(\Sigma) f(E_i(\Sigma))}$$

for $i = I$ and II . For $i = I$ and II , let

$$E_{i, \frac{1}{2}} = \{i' \in E_i(\Sigma) \setminus R^* : i' \in s_{\frac{1}{2}}(A_{\bullet j}) \text{ for some } j \in S_{\frac{1}{2}}\}$$

and $E = \cup_{k=\xi+1}^r C_k \cup E_{I, \frac{1}{2}} \cup E_{II, \frac{1}{2}}$. Moreover, let

$$N(\Sigma) = \begin{pmatrix} A_{E_{I, \frac{1}{2}} \times S_{\frac{1}{2}}} & 0_{|\overline{E_{II, \frac{1}{2}}}| \times |S_{\frac{1}{2}}|} & A_{E \times S_{\frac{1}{2}}} & A_{E \times F} \\ 0_{|\overline{E_{I, \frac{1}{2}}}| \times |S_{\frac{1}{2}}|} & A_{E_{II, \frac{1}{2}} \times S_{\frac{1}{2}}} & & \\ 1_{1 \times |S_{\frac{1}{2}}|} & 0_{1 \times |S_{\frac{1}{2}}|} & 1_{1 \times |S_{\frac{1}{2}}|} & 0_{1 \times |F|} \\ 0_{1 \times |S_{\frac{1}{2}}|} & 1_{1 \times |S_{\frac{1}{2}}|} & 1_{1 \times |S_{\frac{1}{2}}|} & 0_{1 \times |F|} \end{pmatrix},$$

where the rows of $N(\Sigma)$ (except the two last ones) are indexed by the elements in E ($\overline{E_{i, \frac{1}{2}}} = E \setminus E_{i, \frac{1}{2}}$ for $i = I$ and II). See Figure 10.4. Denote by n_I and n_{II} the indexes of the two last rows, respectively.

Lemma 10.4 *Suppose that A has a bicyclic representation $G(A)$. For any non-central cell C_l and a cell $C_{l'}$ for some $1 \leq l \leq \xi$ and $1 \leq l' \leq r$, a node belonging to $T(C_l)$ and $T(C_{l'})$ is either equal to a central node, if $R^* = R_l$, or an endnode of the interval with edge index set R_l otherwise.*

Proof. This follows from the connectivity of A , the construction of the cells and the definition of a central cell. ■

Let us prove the following proposition.

Proposition 10.5 *If the matrix A is bicyclic, then there exists a bicompatible bipartition $\Sigma(\mathcal{K})$ such that for any bipartition $\Sigma'(\mathcal{K})$ equivalent to $\Sigma(\mathcal{K})$, the matrix $M_i(\Sigma')$ is $R_i^*(\Sigma')$ -cyclic for $i = I$ and II and $N(\Sigma')$ is $\{n_I, n_{II}\}$ $\frac{1}{2}$ -binet.*

Proof. Let $G(A)$ be a bicyclic representation of A and $\Sigma(\mathcal{K}) = \{\mathcal{K}_I, \mathcal{K}_{II}\}$ the bipartition induced by $G(A)$. By Lemma 10.3, we deduce that T_I and T_{II} contain each one at most two central cells. Thus the bipartition $\Sigma(\mathcal{K})$ is bicompatible. For any $i \in \{I, II\}$, by deleting all nonbasic edges with index not in $f(E_i(\Sigma))$ and contracting all basic edges with index not in $E_i(\Sigma)$ (in any order), one obtains an $R_i^*(\Sigma)$ -cyclic representation of $M_i(\Sigma)$.

A $\{n_I, n_{II}\}$ $\frac{1}{2}$ -binet representation of $N(\Sigma)$ is obtained as follows. Observe that for $i = I$ and II , $E_{i, \frac{1}{2}}$ is the edge index set a tree rooted at the central node of T_i ; for any 2-edge f_j ($j \in S_{\frac{1}{2}}$), the intersection of this rooted tree with the fundamental circuit of f_j is a directed path denoted by $q_{j,i}$ starting at the central node of T_i ($q_{j,i}$ may be equal to the central node of T_i). For $i = I$ and II , add a nonbasic half-edge entering the terminal node of $q_{j,i}$ for all $j \in S_{\frac{1}{2}}$, and create a basic half-edge n_i entering the central node of T_i . Then, delete all

nonbasic edges (except half-edges) with index not in $F \cup S_{\frac{1}{2}}$, all basic edges with index not in $E \cup \{n_I, n_{II}\}$ and the remaining isolated nodes. (See Figure 10.5.)

Suppose now that \mathcal{K}_I contains at least one non-central cell, say C_l , and \mathcal{K}_{II} at least two, say $C_{l'}$ and $C_{l''}$. Let us prove that there exists a bicyclic representation of A inducing a bicompatible bipartition $\Sigma'(\mathcal{K}) = \{\mathcal{K}'_I, \mathcal{K}'_{II}\}$ of \mathcal{K} with $\mathcal{K}'_I = \mathcal{K}_I \cup \{C_{l''}\}$ and $\mathcal{K}'_{II} = \mathcal{K}_{II} \setminus \{C_{l''}\}$. From the definition of a central cell, it follows that $\cup_{j \in S_{\frac{1}{2}}} s(A_{\bullet j}) \cap C_k = R_k$ and $A_k^{\frac{1}{2} \rightarrow 1}$ is a network matrix for $k = l, l'$ and l'' . For any $1 \leq k \leq r$, let

$$G_k = G(A_{C_k \times (f(C_k) \setminus S_{\frac{1}{2}})}) \subseteq G(A).$$

Assume that $T(C_{l''})$ has only directed edges. Make a copy $\tilde{G}_{l''}$ of $G_{l''}$ and denote by $t_{l''}$ (resp., $u_{l''}$) the first (resp., terminal) node of the copy of $\mathcal{P}_{l''}$ in $\tilde{G}_{l''}$. Suppose that there exists an endnode of \mathcal{P}_l , say u_l , which is not central and such that \mathcal{P}_l enters u_l . Let $\mathcal{T}(u_l)$ be the set of subgraphs G_k in $G(A)$ with $1 \leq k \leq r$ containing u_l . Then contract all edges of $G_{l''}$ and cut the basic cycle in T_l at u_l , by creating a copy \tilde{u}_l of u_l so that $u_l \in G_l$ and $\tilde{u}_l \in G_k$ for all $G_k \in \mathcal{T}(u_l)$ with $k \neq l$. Finally, identify $t_{l''}$ with u_l , and $u_{l''}$ with \tilde{u}_l . Thus, we obtain a new bicyclic representation of A inducing a bipartition $\Sigma'(\mathcal{K})$ equivalent to $\Sigma(\mathcal{K})$. See Figure 10.3 for an example with $C_l = C_3 = \{2, 5\}$, $C_{l'} = \{8, 9\}$ and $C_{l''} = C_5 = \{10\}$.

The other cases can be treated in a similar way by using Lemma 10.4, a network representation of $A_k^{\frac{1}{2} \rightarrow 1}$ for $k = l, l'$ or l'' , and switching operations if necessary. By moving some cells this way, one can prove as above that for any bipartition $\Sigma'(\mathcal{K})$ equivalent to $\Sigma(\mathcal{K})$, the matrix $M_i(\Sigma')$ is $R_i^*(\Sigma')$ -cyclic for $i = I$ and II and $N(\Sigma')$ is $\{n_I, n_{II}\}$ $\frac{1}{2}$ -binet. ■

For computing a $\{n_I, n_{II}\}$ $\frac{1}{2}$ -binet representation of $N(\Sigma)$, we use the following auxiliary matrix:

$$\tilde{N}(\Sigma) = \begin{pmatrix} & A_{E_{I, \frac{1}{2}} \times S_{\frac{1}{2}}} & 0_{|\overline{E_{II, \frac{1}{2}}}| \times |S_{\frac{1}{2}}|} & & \\ & 0_{|\overline{E_{I, \frac{1}{2}}}| \times |S_{\frac{1}{2}}|} & A_{E_{II, \frac{1}{2}} \times S_{\frac{1}{2}}} & 0_{|E| \times 1} & \\ N(\Sigma) & 1_{1 \times |S_{\frac{1}{2}}|} & 1_{1 \times |S_{\frac{1}{2}}|} & 1 & \\ & 1_{1 \times |S_{\frac{1}{2}}|} & 1_{1 \times |S_{\frac{1}{2}}|} & 1 & \end{pmatrix}.$$

Suppose that the matrix $\tilde{N}(\Sigma)$ has a network representation $G(\tilde{N})$. Due to the last column of \tilde{N} the basic edges e_{n_I} and $e_{n_{II}}$ are adjacent. Denote by v_0 the node in $G(\tilde{N})$ incident with e_{n_I} and $e_{n_{II}}$. Observe that for any $j \in S_{\frac{1}{2}}$ and $i \in \{I, II\}$, $(s(A_{\bullet j}) \cap E_{i, \frac{1}{2}}) \cup \{n_i\}$ and $(s(A_{\bullet j}) \cap E_{i, \frac{1}{2}}) \cup \{n_I, n_{II}\}$ are edge index sets of directed paths. Moreover, since A is connected, for all cell C_k with $\xi < k \leq r$ there exists $j' \in S_{\frac{1}{2}}$ such that $s(A_{\bullet j'}) \cap C_k \neq \emptyset$. This implies that the only basic edges incident with v_0 are e_{n_I} and $e_{n_{II}}$.

Procedure Bicyclic(A)

Input: A connected $\frac{1}{2}$ -equisupported matrix A with entries 0, 1, $\frac{1}{2}$ or 2 and at least one $\frac{1}{2}$ -entry.

Output: A bicyclic representation $G(A)$ of A , or determines that none exists.

- 1) compute the quotient set \mathcal{S} ;
- 2) **for** every class $[\Sigma(\mathcal{K})]$ in \mathcal{S} , **do**

- 3) for $i = I$ and II , call $\text{RCyclic}(M_i(\Sigma), R_i^*(\Sigma))$ of Section 8.2;
if we have an $R_i^*(\Sigma)$ -cyclic representation $G(M_i)$ of $M_i(\Sigma)$ for $i = I$ and II ,
then go to 4, otherwise return to 2;
 - 4) compute a network representation $G(\tilde{N})$ of $\tilde{N}(\Sigma)$, if one exists, otherwise return to 2,
delete v_0 and using switching operations, build up a $\{n_I, n_{II}\}$ $\frac{1}{2}$ -binet representation
 $G(N)$ of $N(\Sigma)$ such that e_{n_I} and $e_{n_{II}}$ are entering;
 - 5) contract all half-edges in $G(M_I)$, $G(M_{II})$ and $G(N)$; and for $i = I$ and II , identify
the subtree of $G(N)$ with edge index set $E_{i, \frac{1}{2}}$ with the corresponding tree
in $G(M_i)$ by keeping the labeling of the basic edges in $G(M_i)$;
output the resulting bicyclic representation $G(A)$ of A and STOP;
- endfor**
output that A is not bicyclic;

Finally, we prove Theorems 10.1 and 10.2.

Proof of Theorems 10.1 and 10.2. The "only if" part of Theorem 10.2 follows from Proposition 10.5.

Let us prove the counter part of Theorem 10.2 and the correctness of the procedure Bicyclic. Suppose that there exists a bicompatible bipartition $\Sigma_0(\mathcal{K}) = \{\mathcal{K}_I, \mathcal{K}_{II}\}$ such that for any bipartition $\Sigma(\mathcal{K})$ equivalent to $\Sigma_0(\mathcal{K})$, the matrix $M_i(\Sigma)$ is $R_i^*(\Sigma)$ -cyclic for $i = I$ and II and $N(\Sigma)$ is $\{n_I, n_{II}\}$ $\frac{1}{2}$ -binet. Suppose that the procedure Bicyclic is dealing with the class $[\Sigma_0(\mathcal{K})]$ in step 2. By Theorem 8.1, for some bipartition Σ equivalent to Σ_0 , the procedure RCyclic computes an $R_i^*(\Sigma)$ -cyclic representation $G(M_i)$ of $M_i(\Sigma)$ for $i = I$ and II , and a $\{n_I, n_{II}\}$ $\frac{1}{2}$ -binet representation $G(N)$ of $N(\Sigma)$ is computed. Since A is connected, a cell C_k such that $R_k = \emptyset$ ($\xi < k \leq r$) necessarily intersects the support of a column with index in $S_{\frac{1}{2}}$, and for each $j \in F$, $s(A_{\bullet j})$ is the edge index set of a path in $G(N)$. We observe that $(A_{E_{I, \frac{1}{2}} \times S_{\frac{1}{2}}}^T \ 0_{|\overline{E_{I, \frac{1}{2}}}| \times |S_{\frac{1}{2}}|}^T \ 1_{1 \times |S_{\frac{1}{2}}|}^T \ 0_{1 \times |S_{\frac{1}{2}}|}^T)^T$ is a submatrix of $N(\Sigma)$, and $s(A_{\bullet j}) \cap E_{I, \frac{1}{2}}$ is the edge index set of a directed path in $G(M_I)$ starting at the central node, for all $j \in S_{\frac{1}{2}}$. Thus, the basic subtrees of $G(N)$ and $G(M_I)$ with edge index set $E_{I, \frac{1}{2}}$ are isomorphic; the former is rooted at the endnode of e_{n_I} while the latter is rooted at the central node of $G(M_I)$. The same thing holds with $G(M_{II})$, $E_{II, \frac{1}{2}}$ and $e_{n_{II}}$ instead of $G(M_I)$, $E_{I, \frac{1}{2}}$ and e_{n_I} , respectively. We deduce that the procedure Bicyclic outputs a bicyclic representation of A (see Figure 10.5 for an example).

For the proof of Theorem 10.1, it remains to show that the procedure Bicyclic works in time $O(nm\alpha)$. In a bicompatible bipartition $\Sigma(C) = \{\mathcal{K}_I, \mathcal{K}_{II}\}$, since any \mathcal{K}_i ($i = I$ or II) has at most 2 central cells, there are at most 6 different ways of partitioning the central cells into two groups. Furthermore, any bipartition of the central cells can be extended to at most 3 nonequivalent bicompatible partitions of \mathcal{K} . Therefore, the number of passages through step 1 is upper bounded by 18. By Theorem 2.5, the computational effort required to obtain a network representation of a matrix $A_k^{\frac{1}{2} \rightarrow 1}$ for some k is $O(|C_k| \alpha_k)$ where α_k is the number of nonzero elements in A_k . Thus computing the quotient set \mathcal{S} can be executed in time $O(n\alpha)$. Similarly, step 4 takes time $O(n\alpha)$. Finally, by Theorem 8.1, the procedure RCyclic works in time $O(nm\alpha)$. This completes the proof. \blacksquare

Chapter 11

Recognizing $\{\epsilon, \rho\}$ -central matrices

In this chapter, we provide a characterization of $\{\epsilon, \rho\}$ -central (non-network) $\{0, 1, 2\}$ -matrices, where ϵ and ρ are two given row indexes, as well as a recognition procedure. Let A be a $\{0, 1, 2\}$ -matrix of size $n \times m$ and α the number of nonzero entries in A . We assume that A is not a network matrix. Let ϵ and ρ ($\rho \neq 1$) be two row indexes and $S_0 = \{j : \epsilon, \rho \in s(A_{\bullet j})\}$. We will prove the following.

Theorem 11.1 *Suppose $S_0 = \emptyset$. The matrix A can be tested for being $\{\epsilon, \rho\}$ -central by the procedure CentralI. The running time of the procedure is $O(n^3\alpha)$.*

Theorem 11.2 *Suppose $S_0 \neq \emptyset$. The matrix A can be tested for being $\{\epsilon, \rho\}$ -central by the procedure CentralII. The running time of the procedure is $O(n^3\alpha)$.*

A characterization of cyclic matrices without $\frac{1}{2}$ - nor 2-entry turns out to be the most critical part in the recognition of binet matrices. Provided that A is cyclic, there is a priori no direct way of detecting the edge index set of the basic cycle, in some cyclic representation of A . To illustrate this, consider the cyclic matrix in Figure 11.1. Its minimally non-network submatrices are:

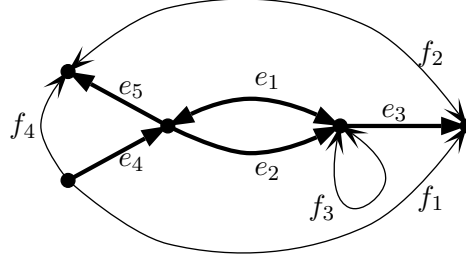
$$M_1 = \begin{array}{c|ccc} & f_1 & f_2 & f_3 \\ \hline e_1 & 0 & 1 & 1 \\ e_2 & 1 & 0 & 1 \\ e_3 & 1 & 1 & 0 \end{array} \quad M_2 = \begin{array}{c|ccc} & f_1 & f_2 & f_4 \\ \hline e_3 & 1 & 1 & 0 \\ e_4 & 1 & 0 & 1 \\ e_5 & 0 & 1 & 1 \end{array}$$

By careful analysis, one can show that there is no cyclic representation of A whose basic cycle has an edge index set equal to the row index set of M_1 or M_2 . However, given the row indexes 1 and 2 and since A is nonnegative, we describe a procedure for computing a $\{1, 2\}$ -central representation of A .

The recognition of $\{\epsilon, \rho\}$ -central matrices can be viewed as a generalization of Schrijver's method outlined on page 31. Instead of considering a row index i , and then carrying on with the connected components of the matrix $A_{\overline{\{i\}} \times \overline{f(\{i\})}}$, we fix two row indexes, namely ϵ and ρ , and deal with the connected components of the matrix $A_{\overline{\{\epsilon, \rho\}} \times \overline{f(\{\epsilon, \rho\})}}$.

The chapter is organised as follows. First, we give the main notations with graphical interpretations and state two theorems characterizing $\{\epsilon, \rho\}$ -central matrices in case $S_0 = \emptyset$ (respectively, $S_0 \neq \emptyset$) requiring certain assumptions that are discussed in Section 11.1. Then,

	f_1	f_2	f_3	f_4
e_1	0	1	1	0
e_2	1	0	1	0
e_3	1	1	0	0
e_4	1	0	0	1
e_5	0	1	0	1

Figure 11.1: A cyclic matrix A and a $\{1, 2\}$ -central representation of A .

in Sections 11.2 and 11.3, we deal with the proof of the first and second theorem, respectively. Section 11.4 provides a procedure for recognizing $\{1, \rho\}$ -noncorelated network matrices, that is used in Section 11.2.

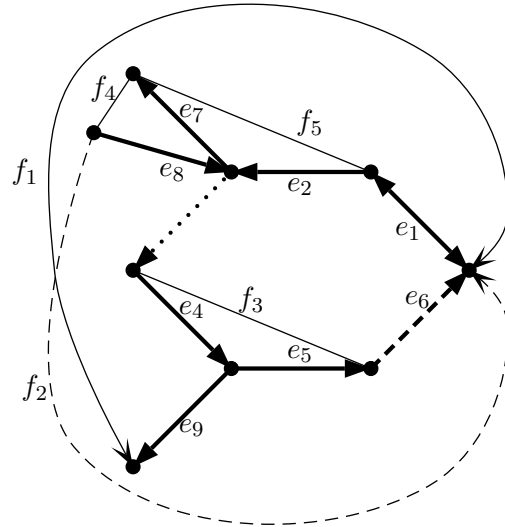
11.1 Preliminaries

Let ϵ and $\rho \neq 1$ be two row indexes of A , $R^* = \{\epsilon, \rho\}$ and $D = (V, \Upsilon)$ be a digraph with respect to R^* , as computed in Section 7.3. We will make use of the whole terminology described in Chapter 7. Let $S^* = \{j : s(A_{\bullet j}) \cap R^* \neq \emptyset\}$, $S_0 = \{j \in S^* : \epsilon, \rho \in s(A_{\bullet j})\}$, $S_1 = \{j \in S^* : \epsilon \in s(A_{\bullet j}), \rho \notin s(A_{\bullet j})\}$ and $S_2 = \{j \in S^* : \epsilon \notin s(A_{\bullet j}), \rho \in s(A_{\bullet j})\}$. For any bonsai E_ℓ and $k \in \{0, 1, 2\}$, we define the *connector set* $f_{S_k}(E_\ell) = \{j \in S_k : s(A_{\bullet j}) \cap E_\ell \neq \emptyset\}$. For any $j \in S^*$ and $1 \leq \ell \leq b$, denote $E_{\ell j} = E_\ell \cap s(A_{\bullet j})$, and for $S \subseteq S_1 \cup S_2$, $E_{\ell S} = \cup_{j \in S} E_{\ell j}$. Throughout this section we assume $\epsilon = 1$.

$$S_1 = \{1\}; \quad S_2 = \{2\};$$

$$E_1 = \{2; 7; 8\}; \quad E_2 = \{4; 5\}; \quad E_3 = \{9\};$$

$\ell =$	1	2	3
$f_{S_1}(E_\ell)$	$\{1\}$	$\{1\}$	$\{1\}$
$f_{S_2}(E_\ell)$	$\{2\}$	$\{2\}$	\emptyset
$E_{\ell S_1}$	$\{2\}$	$\{4\}$	$\{9\}$
$E_{\ell S_2}$	$\{8\}$	$\{4, 5\}$	\emptyset

Figure 11.2: a $\{1, 6\}$ -central representation of some binet $\{0, 1\}$ -matrix, where E_1 is disjointly shared, E_2 is jointly shared and E_3 is S_1 -dominated. (f_3, f_4 and f_5 are directed edges.)

$$S_0 = \{3, 4\}; \quad S_1 = \{1\}; \quad S_2 = \{2\};$$

$$E_1 = \{7; 8\}; \quad E_2 = \{4\}; \\ E_3 = \{9\}; \quad E_4 = \{10\}; \quad E_5 = \{11\};$$

$\ell =$	1	2	3, 4, 5
$f_{S_0}(E_\ell)$	\emptyset	$\{3, 4\}$	$\{3, 4\}$
$f_{S_1}(E_\ell)$	$\{1\}$	\emptyset	\emptyset
$f_{S_2}(E_\ell)$	$\{2\}$	$\{2\}$	\emptyset
$E_{\ell S_1}$	$\{7\}$	\emptyset	\emptyset
$E_{\ell S_2}$	$\{8\}$	$\{4\}$	\emptyset

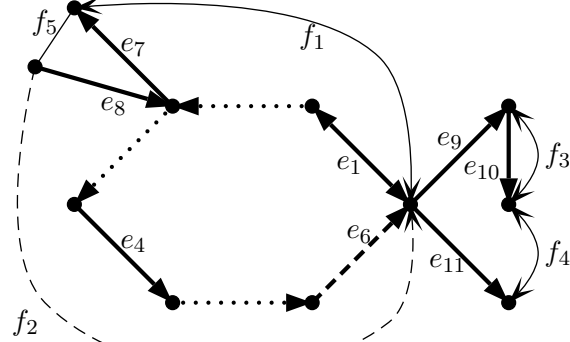


Figure 11.3: a (partial) $\{1, 6\}$ -central representation of some binet $\{0, 1, 2\}$ -matrix, where E_1 is sensitive, E_2 and E_4 are S_0 -straight, $E_3 \in V_2$ and $E_5 \in V_0 - (V_2 \cup V_{st})$. (f_5 is directed.)

We say that a bonsai $E_\ell \in V$ ($1 \leq \ell \leq b$) is S_k -dominated if $f_{S_k}(E_\ell) \neq \emptyset$ and $f_{S_{k'}}(E_\ell) = \emptyset$ for some $k, k' \in \{1, 2\}$ ($k \neq k'$). A bonsai $E_\ell \in V$ is *shared* if $f_{S_1}(E_\ell) \neq \emptyset$ and $f_{S_2}(E_\ell) \neq \emptyset$. A shared bonsai $E_\ell \in V$ is said to be *disjointly shared* if $E_{\ell S_1} \cap E_{\ell S_2} = \emptyset$, otherwise *jointly shared*. See Figure 11.2.

	f_1	f_2	f_4	
e_7	1	0	1	0
e_8	0	1	1	0
\tilde{e}_1	1	0	0	1
\tilde{e}_2	0	1	0	1

Table 11.1: The bonsai matrix N_1 with respect to E_1 given in Figure 11.3.

A bonsai $E_\ell \in V$ is called *sensitive*, if it is disjointly shared, $f_{S_0}(E_\ell) = \emptyset$, $J_\ell^1 = \{E_{\ell j} : j \in f_{S_k}(E_\ell)\}$ and $J_\ell^2 = \{E_{\ell j} : j \in f_{S_{k'}}(E_\ell)\}$ for some $k, k' \in \{1, 2\}$ $k \neq k'$, and N_ℓ is a network matrix (see page 90 for the definition of J_ℓ^1 and J_ℓ^2). For every $1 \leq \ell \leq b$, the bonsai matrix N_ℓ is defined in Section 7.2. For instance, the bonsai E_1 in Figure 11.2 is disjointly shared but not sensitive, because $J_1^1 = \emptyset$, while E_1 in Figure 11.3 is sensitive (see Table 11.1). For $1 \leq \ell, \ell' \leq b$, we say that E_ℓ and $E_{\ell'}$ are S_k linked, or E_ℓ is S_k linked to $E_{\ell'}$, if $f_{S_k}(E_\ell) \cap f_{S_k}(E_{\ell'}) \neq \emptyset$ for some $k \in \{1, 2\}$. (If E_ℓ and $E_{\ell'}$ are S_1 and S_2 linked, then we write that E_ℓ and $E_{\ell'}$ are S_1, S_2 linked.) As an example, in Figure 11.2, E_1 and E_2 are S_1, S_2 linked.

Let $V_0 = \{E_\ell : f_{S_0}(E_\ell) \neq \emptyset\}$, $V'_0 = V_0 \cup \{\text{shared bonsai}\}$, $V_2 = \{E_\ell : \exists \beta \in S_0 \text{ with } g_\beta(E_\ell) = 2\}$ (see page 94 for the definition of g). A bonsai E_ℓ is said to be S_0 -straight if $E_\ell \in V_0 - V_2$ and $E_{\ell\beta} = E_{\ell\beta'}$ for any $\beta, \beta' \in S_0$. Let V_{st} denote the set of S_0 -straight bonsais. In Figure 11.3, E_2 and E_4 are S_0 -straight, while $E_3 \in V_2$ since $g_3(E_3) = 2$ (see Proposition 7.12), and E_5 is neither in V_2 nor S_0 -straight.

For a set $V' \subseteq V$, we define the undirected graph $F^*(V')$ whose vertex set is V' , and two vertices E_ℓ and $E_{\ell'}$ are adjacent if and only if $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$, $(E_\ell, E_{\ell'}) \notin D$ and $(E_{\ell'}, E_\ell) \notin D$. Provided that A has a $\{1, \rho\}$ -central representation $G(A)$, if some bonsai $E_\ell \in V$ has some property (for instance E_ℓ is sensitive), then we say that the corresponding bonsai B_ℓ in $G(A)$ has this property (for instance B_ℓ is sensitive), and conversely.

Before explaining the graphical motivations of these definitions, we state the main theorems. Later, we will give the definition of left-compatibility for a set $U \subseteq V$ of bonsais. For any left-compatible set $U \subseteq V$, we will define a U -spanning pair (j_1, j_2) of column indexes as well as a subset $V(j_1, j_2)$ of V , and we will describe an instance $\Omega(U, j_1, j_2)$ of the 2-SAT problem, where variables are associated to bonsais in $V(j_1, j_2)$. We will also need the following.

assumption \mathcal{A} : for any sensitive bonsai E_ℓ , if some bonsai $E_{\ell'}$ is S_1, S_2 linked to E_ℓ , then $J_{\ell'}^2 = \emptyset$ and $N_{\ell'}$ is a network matrix.

Under this assumption, we shall prove the following theorems.

Theorem 11.3 *Suppose $S_0 = \emptyset$. Then the matrix A is $\{1, \rho\}$ -central if and only if $g_\beta(E_\ell) \leq 1$ for all $1 \leq \ell \leq b$, $\beta \in f^*(E_\ell)$, and there exists a left-compatible set U such that for any U -spanning pair (j_1, j_2) , the graph $F^*(\overline{V(j_1, j_2)})$ is bipartite, for all $E_\ell \in \overline{V(j_1, j_2)}$ the bonsai matrix N_ℓ is a network matrix and $J_\ell^2 = \emptyset$, and the instance $\Omega(U, j_1, j_2)$ of the 2-SAT problem has a truth assignment.*

Theorem 11.4 *Suppose $S_0 \neq \emptyset$. Then the matrix A is $\{1, \rho\}$ -central if and only if the graph $F^*(\overline{V_0})$ is bipartite, for any $E_\ell \in \overline{V_0}$ the bonsai matrix N_ℓ is a network matrix and $J_\ell^2 = \emptyset$, and there exists a right-compatible set U such that the instance $\Lambda(U)$ of the 2-SAT problem has a truth assignment.*

Suppose that A has a basic $\{1, \rho\}$ -central representation $G(A)$. Up to row permutations and changing the value of ρ , we may assume that w_1, \dots, w_ρ are the vertices of the basic cycle in this order, $e_1 = [w_1, w_\rho]$ and $e_\rho = [w_{\rho-1}, w_\rho] \in G(A)$. Hence w_ρ is a central node. Recall that T denotes the basic maximal 1-tree in $G(A)$, $G_0(A)$ the connected component of $T - \{e_1, e_\rho\}$ containing w_1 (and $w_{\rho-1}$) and $G_1(A)$ the other connected component. We say that a subgraph of $G_0(A)$ and $G_1(A)$ is *on the left* and *on the right*, respectively, of $\{e_1, e_\rho\}$. For all $1 \leq \ell \leq b$, the bonsai E_ℓ is the edge index set of a tree in $G(A)$ denoted as B_ℓ and called a *bonsai* in $G(A)$.

For any w_i and $w_{i'}$ on the basic cycle with $1 \leq i, i' \leq \rho - 1$, let $p(w_i, w_{i'})$ be the basic path in $G_0(A)$ between w_i and $w_{i'}$, and $I(w_i, w_{i'})$ its row index set. For any two bonsais B_ℓ and $B_{\ell'}$ intersecting the basic cycle, if $i \leq i'$ for all $w_i \in B_\ell$ and $w_{i'} \in B_{\ell'}$ ($1 \leq i, i' \leq \rho - 1$), then B_ℓ is *preceding* $B_{\ell'}$ and $B_{\ell'}$ is *succeeding* B_ℓ . As an example, in Figure 11.3, B_1 is preceding B_2 . If a bonsai B_ℓ is on the left of $\{e_1, e_\rho\}$, then denote by $v_{\ell,1}$ (respectively, $v_{\ell,2}$) the node of B_ℓ which is the closest to w_1 (respectively, $w_{\rho-1}$) in $G_0(A)$. If a bonsai B_ℓ is on the right of $\{e_1, e_\rho\}$, we denote by v_ℓ the closest vertex of B_ℓ to the central node w_ρ (v_ℓ may be equal to w_ρ).

For a bonsai E_ℓ , under certain conditions and provided that A is $\{1, \rho\}$ -central, a subset $E_{\ell j}$ of E_ℓ , with $j \in f^*(E_\ell)$, can be interpreted as the edge index set of a path.

Lemma 11.5 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. Let $j \in f_{S_k}(E_\ell)$ for some $1 \leq \ell \leq b$ and $k \in \{0, 1, 2\}$. If $k \in \{1, 2\}$, or $k = 0$ and B_ℓ is S_0 -straight, then the following holds:*

- (i) *The set $E_{\ell j}$ is the edge index set of the unique B_ℓ -path generated by f_j , and $g_j(E_\ell) = 1$.*
- (ii) *If B_ℓ is on the left of $\{e_1, e_\rho\}$, then the B_ℓ -path generated by f_j leaves $v_{\ell,1}$ for $k = 1$, and enters $v_{\ell,2}$ for $k = 2$.
If B_ℓ is on the right of $\{e_1, e_\rho\}$, then the B_ℓ -path generated by f_j leaves v_ℓ .*

Proof. If $k \in \{1, 2\}$, then by Corollary 3.6, the basic fundamental circuit of f_j is a non-closed path. Hence, f_j generates exactly one B_ℓ -path with edge index set $E_{\ell j}$ and, using Lemma 4.1, $s_2(A_{\bullet j}) = \emptyset$. So, by Proposition 7.12, $g_j(E_\ell) = 1$. Similarly, if $k = 0$ and B_ℓ is S_0 -straight, since $g_j(E_\ell) = 1$, by Proposition 7.12, $E_\ell^{II}(A_{\bullet j}) = \emptyset$ and $E_\ell^I(A_{\bullet j}) = E_{\ell j}$ is the edge index set of the unique B_ℓ -path generated by f_j . Then part (ii) is implied by Lemma 4.2 since A is nonnegative, and the fact that $e_1 = [w_1, w_\rho]$ and $e_\rho =]w_{\rho-1}, w_\rho]$. ■

Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. By Lemma 11.5, for all $1 \leq \ell \leq b$ and $j \in f_{S_k}(E_\ell)$, if $k \in \{1, 2\}$, or $k = 0$ and B_ℓ is S_0 -straight, then $E_{\ell j}$ is the edge index set of a B_ℓ -path, denoted as $B_{\ell j}$.

Lemma 11.6 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. Let B_ℓ be a bonsai on the left of $\{e_1, e_\rho\}$. Then*

- (i) *$E_{\ell S_k}$ is the edge index set of an out-rooted (resp., in-rooted) tree for $k = 1$ (resp., $k = 2$).*
- (ii) *For all $j \in S_0$, $E_{\ell j}$ is the edge index set of edges belonging to B_ℓ and the basic cycle.*

Proof. The proof of part (i) directly follows from Lemma 11.5. Now let $j \in f_{S_0}(E_\ell)$. Since A is nonnegative, by Corollary 3.6 and Lemma 7.17, the fundamental circuit of f_j is a handcuff. Then, by Lemma 4.2, it follows that the fundamental circuit of f_j intersected with $G_0(A)$ is equal to the basic path from w_1 to $w_{\rho-1}$ (contained in the basic cycle. this concludes the proof. ■

Provided that A has a $\{1, \rho\}$ -central representation $G(A)$, by Lemma 11.6 (i), we denote by $T_{\ell,k}$ the subgraph of $G(A)$ with edge index set $E_{\ell S_k}$ (see Figure 11.4).

Let us see now a first enlightenment on sensitive bonsais.

Lemma 11.7 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. Let B_ℓ be a shared bonsai on the left of $\{e_1, e_\rho\}$ in $G(A)$. Then we have*

- *If $v_{\ell,1} = v_{\ell,2}$, then B_ℓ is sensitive and $v_{\ell,1}$ is the unique node of B_ℓ belonging to the basic cycle.*
- *If $v_{\ell,1} \neq v_{\ell,2}$, then the path from $v_{\ell,1}$ to $v_{\ell,2}$ in B_ℓ lies on the basic cycle.*

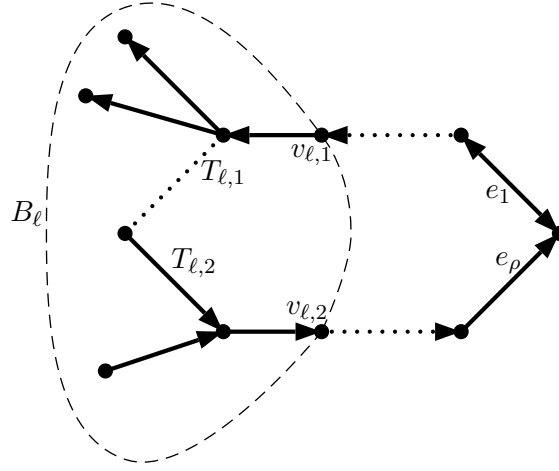


Figure 11.4: An illustration of the trees $T_{\ell,1}$ and $T_{\ell,2}$ whose edge index sets are $E_{\ell S_1}$ and $E_{\ell S_2}$, respectively, for some bonsai E_ℓ .

Proof. Assume first that $v_{\ell,1} = v_{\ell,2}$. Suppose, to the contrary, that $v_{\ell,1}$ does not belong to the basic cycle. Let $j_1 \in f_{S_1}(E_\ell)$, $j_2 \in f_{S_2}(E_\ell)$ and e_i be a basic edge on the basic path from $v_{\ell,1}$ to the basic cycle. By Lemma 11.5, the B_ℓ -paths generated by f_{j_1} and f_{j_2} are leaving and entering $v_{\ell,1}$, respectively. Then, since e_i is in the fundamental circuit of f_{j_1} and f_{j_2} and using Lemma 4.2, we get a contradiction. Hence $v_{\ell,1}$ is on the basic cycle. From the definition of $v_{\ell,1}$ and $v_{\ell,2}$, it follows that $v_{\ell,1}$ is the unique node of B_ℓ belonging to the basic cycle. So by Lemma 11.6 (ii), $f_{S_0}(E_\ell) = \emptyset$. Furthermore, using Lemmas 11.5 and 7.7, we deduce that $\{E_{\ell j} : j \in f_{S_1}(E_\ell)\} \subseteq J_\ell^1$ and $\{E_{\ell j} : j \in f_{S_2}(E_\ell)\} \subseteq J_\ell^2$ or conversely, and B_ℓ is a $v_{\ell,1}$ -rooted network representation of the bonsai matrix N_ℓ . Thus B_ℓ is sensitive.

Now assume $v_{\ell,1} \neq v_{\ell,2}$. Since B_ℓ is connected, there exists a (basic) path in B_ℓ from $v_{\ell,1}$ to $v_{\ell,2}$. Observe also that there exist a basic path from $v_{\ell,1}$ to w_1 and an other from $v_{\ell,2}$ to $w_{\rho-1}$ in $G_0(A)$ that do not intersect and do not contain any edge of B_ℓ . This completes the proof. ■

To make the assumption \mathcal{A} , we will need Lemmas 11.9, 11.10 and 11.11 below. In the case where this assumption is not satisfied, a matrix A' is obtained by adding some columns to A , and we consider a digraph D' with respect to A' . By Lemma 11.10, it turns out that solving the recognition problem on A is equivalent to solving it on A' , and D' has one bonsai less than D . By adding at most m columns to A , the assumption \mathcal{A} will be satisfied. Let us see an auxiliary lemma.

Lemma 11.8 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. Let $1 \leq \ell \leq b$. If B_ℓ is on the right of $\{e_1, e_\rho\}$ in $G(A)$, then $J_\ell^2 = \emptyset$ and N_ℓ is a network matrix.*

Proof. Using Lemma 4.2, it follows that all B_ℓ -paths leave v_ℓ . Hence, by Lemma 7.7, $J_\ell^2 = \emptyset$. Moreover, B_ℓ is a v_ℓ -rooted network representation of N_ℓ . ■

Lemma 11.9 *Suppose that the matrix A has a $\{1, \rho\}$ -central representation $G(A)$. Let B_ℓ be a sensitive bonsai ($1 \leq \ell \leq b$). Then the following holds.*

- 1) The bonsai B_ℓ is on the left of $\{e_1, e_\rho\}$.
- 2) If $v_{\ell,1} \neq v_{\ell,2}$, then for any shared bonsai $B_{\ell'}$ on the left of $\{e_1, e_\rho\}$, we have $v_{\ell',1} = v_{\ell',2} \in B_\ell$ and $B_{\ell'}$ is S_1, S_2 linked to B_ℓ .

Proof. If B_ℓ is on the right of $\{e_1, e_\rho\}$, then by Lemma 11.8 $J_\ell^2 = \emptyset$, contradicting the definition of a sensitive bonsai. Thus B_ℓ is on the left of $\{e_1, e_\rho\}$.

Now assume that $v_{\ell,1} \neq v_{\ell,2}$. Let $B_{\ell'}$ be a shared bonsai on the left of $\{e_1, e_\rho\}$. By Lemma 11.7, the nodes $v_{\ell,1}$, $v_{\ell,2}$, $v_{\ell'}^1$ and $v_{\ell'}^2$ belong to the basic cycle. Suppose, by contradiction, that for some $k \in \{1, 2\}$, $v_{\ell'}^k = w_{i_0}$ is such that $i_0 \geq i$ for all $w_i \in B_\ell$. This implies that $I(v_{\ell,1}, v_{\ell,2}) = E_{\ell_j}$ for all $j \in f_{S_1}(E_{\ell'})$. Let $j_1 \in f_{S_1}(E_{\ell'})$ and $j_2 \in f_{S_2}(E_\ell)$. Since $E_{\ell_{j_1}} = I(v_{\ell,1}, v_{\ell,2})$, the B_ℓ -path $B_{\ell_{j_1}}$ enters $v_{\ell,2}$, and by Lemma 11.5 (ii) $B_{\ell_{j_2}}$ enters $v_{\ell,2}$. So, since N_ℓ is a network matrix, by Lemma 7.10, $E_{\ell_{j_1}} \sim_{E_\ell} E_{\ell_{j_2}}$, contradicting the definition of B_ℓ . One gets a similar contradiction if $i_0 \leq i$ for all $w_i \in B_\ell$. It results that $v_{\ell',1}, v_{\ell',2} \in B_\ell$ and $v_{\ell',k} \neq v_{\ell,1}, v_{\ell,2}$ for $k = 1$ and 2 . Since $E_\ell \cap E_{\ell'} = \emptyset$, $v_{\ell',1} = v_{\ell',2}$. At last, as $v_{\ell',1}$ is an inner node on the path $p(v_{\ell,1}, v_{\ell,2})$ and $B_{\ell'}$ is shared, the bonsais B_ℓ and $B_{\ell'}$ are S_1, S_2 linked. ■

Lemma 11.10 *Let E_ℓ be a sensitive bonsai and $E_{\ell'}$ some bonsai S_1, S_2 linked to E_ℓ such that $J_{\ell'}^2 \neq \emptyset$ or $N_{\ell'}$ is not a network matrix. Then for any $j_1 \in f_{S_1}(E_\ell) \cap f_{S_1}(E_{\ell'})$, the matrix A is $\{1, \rho\}$ -central if and only if the matrix $A' = [A \chi_{E_{\ell_{j_1}} \cup E_{\ell'_{j_1}}}^{\{1, \dots, n\}}]$ is $\{1, \rho\}$ -central.*

Proof. Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. Since $J_{\ell'}^2 \neq \emptyset$ or $N_{\ell'}$ is not a network matrix, Lemma 11.8 implies that $B_{\ell'}$ is on the left of $\{e_1, e_\rho\}$. As $B_{\ell'}$ is S_1, S_2 linked to B_ℓ , $B_{\ell'}$ is shared. So, if $v_{\ell,1} \neq v_{\ell,2}$, then by Lemma 11.9 $v_{\ell',1} = v_{\ell',2} \in B_\ell$. Suppose now that $v_{\ell,1} = v_{\ell,2}$. Since B_ℓ and $B_{\ell'}$ are shared, by Lemma 11.7, $v_{\ell,1} = w_{i_0}$ for some $2 \leq i_0 \leq \rho - 1$, and $v_{\ell',1}$ and $v_{\ell',2}$ lie on the basic cycle. Hence, if $i_0 \geq i$ for all $w_i \in B_{\ell'}$, then $f_{S_2}(E_\ell) \cap f_{S_2}(E_{\ell'}) = \emptyset$, which contradicts the fact that B_ℓ and $B_{\ell'}$ are S_1, S_2 linked. If $i_0 \leq i$ for all $w_i \in B_{\ell'}$, then $f_{S_1}(E_\ell) \cap f_{S_1}(E_{\ell'}) = \emptyset$ and we get a same contradiction. Therefore w_{i_0} is an inner node on the path $p(v_{\ell,1}, v_{\ell,2}) \subseteq B_{\ell'}$. Whatever we have $v_{\ell',1} = v_{\ell',2} \in B_\ell$ or $v_{\ell,1} = v_{\ell,2} \in B_{\ell'}$, it yields that $E_{\ell_{j_1}} \cup E_{\ell'_{j_1}}$ is the edge index set of a path in $G(A)$ for any $j_1 \in f_{S_1}(E_\ell) \cap f_{S_1}(E_{\ell'})$. Therefore A' is $\{1, \rho\}$ -central. The converse part is trivial. ■

Lemma 11.11 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. Let B_ℓ be a sensitive bonsai and $B_{\ell'}$ some bonsai S_1, S_2 linked to B_ℓ . Then, $B_{\ell'}$ is on the left of $\{e_1, e_\rho\}$ if and only if $J_{\ell'}^2 \neq \emptyset$ or $N_{\ell'}$ is not a network matrix.*

Proof. The proof of the "if" part follows from Lemma 11.8.

Suppose that $B_{\ell'}$ is on the left of $\{e_1, e_\rho\}$. Since $B_{\ell'}$ is S_1, S_2 linked to B_ℓ , $B_{\ell'}$ is shared. So, if $v_{\ell,1} \neq v_{\ell,2}$, then by Lemma 11.9 $v_{\ell',1} = v_{\ell',2}$. And Lemma 11.7 implies that $B_{\ell'}$ is sensitive. In particular, $J_{\ell'}^2 \neq \emptyset$.

Now assume $v_{\ell,1} = v_{\ell,2}$. By Lemma 11.7 $v_{\ell,1} = w_{i_0}$ for some $2 \leq i_0 \leq \rho - 1$. One can prove in the same way as in the proof of Lemma 11.10 that w_{i_0} is an inner node on the path $p(v_{\ell'}^1, v_{\ell'}^2)$ and $v_{\ell'}^1 \neq v_{\ell'}^2$ (otherwise B_ℓ and $B_{\ell'}$ would not be S_1, S_2 linked). Thus by Lemma 4.2, for any $k \in \{1, 2\}$ and $j_k \in f_{S_k}(E_\ell) \cap f_{S_k}(E_{\ell'})$, the B_ℓ -paths $B_{\ell_{j_1}}$ and $B_{\ell_{j_2}}$ leave and enter

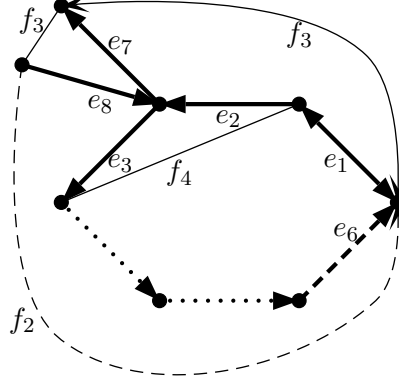


Figure 11.5: an example of a (partial) $\{1,6\}$ -central representation of some binet matrix, where $E_1 = \{7,8\}$ and $E_2 = \{2,3\}$ are sensitive, $v_{2,1} \neq v_{2,2}$, and E_1 and E_2 are S_1, S_2 linked.

$v_{\ell,1} = v_{\ell,2}$, respectively; then it results that the $B_{\ell'}$ -paths $B_{\ell'j_1}$ and $B_{\ell'j_2}$ enter and leave $v_{\ell,1}$, respectively. If $N_{\ell'}$ is a network matrix, then by Lemma 7.10 $E_{\ell'j_1} \approx_{E_{\ell'}} E_{\ell'j_2}$, so $J_{\ell'}^2 \neq \emptyset$. This completes the proof. ■

The advantage of assumption \mathcal{A} is revealed in the next lemma.

Lemma 11.12 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$ and assumption \mathcal{A} is satisfied. Then, for any sensitive bonsai B_{ℓ} , there is no bonsai on the left of $\{e_1, e_{\rho}\}$ that is S_1, S_2 linked to B_{ℓ} .*

Proof. This directly follows from assumption \mathcal{A} and Lemma 11.11. ■

In Figure 11.5, there is an example of a central representation of some binet matrix such that assumption \mathcal{A} is not satisfied. The following procedure is the first subroutine of the procedures Centrall and CentralII. It provides a matrix A' of size $n \times m'$ ($m' \geq m$) such that $A'_{\bullet\{1,\dots,m\}} = A$ and satisfying assumption \mathcal{A} .

Procedure Initialization($A, \{\epsilon, \rho\}$)

Input: A non-network matrix A and two row indexes ϵ and $\rho \neq 1$.

Output: A matrix A' such that A' is $\{1, \rho\}$ -central if and only if A is $\{\epsilon, \rho\}$ -central, and assumption \mathcal{A} is satisfied for A' .

- 1) let $A' = A$, permute the row of A' labeled ϵ and the first one;
- 2) let $R^* = \{1, \rho\}$ and partition $\overline{R^*}$ into E_1, \dots, E_b as described in Section 7.3;
- 3) **while** there exist $1 \leq \ell, \ell' \leq b$ such that E_{ℓ} is sensitive, $E_{\ell'}$ is S_1, S_2 linked to E_{ℓ} , and $J_{\ell'}^2 \neq \emptyset$ or $N_{\ell'}$ is not a network matrix, **do**
 choose $j_1 \in f_{S_1}(E_{\ell}) \cap f_{S_1}(E_{\ell'})$ and let $A' = [A' \chi_{E_{\ell j_1} \cup E_{\ell' j_1}}^{\{1, \dots, n\}}]$;
 recompute the partition of $\overline{R^*}$ into E_1, \dots, E_b with respect to A' ;
endwhile
 output the matrix A' ;

Lemma 11.13 *The output of the procedure Initialization is correct. The output matrix A' has at most $2m$ columns.*

Proof. Clearly, after each passage through step 3, the number of vertices in D decreases by 1. Moreover, if some bonsai E_ℓ is sensitive and $E_{\ell'}$ is such that $J_{\ell'}^2 \neq \emptyset$ or $N_{\ell'}$ is not a network matrix, then $|E_\ell| \geq 2$ and $|E_{\ell'}| \geq 2$, which implies $|\bar{f}^*(E_\ell)| \geq 1$ and $|\bar{f}^*(E_{\ell'})| \geq 1$. So the number of columns added to A does not exceed m . By step 3, the assumption \mathcal{A} is satisfied for A' , and by step 1 and Lemma 11.10, A' is $\{1, \rho\}$ -central if and only if A is $\{\epsilon, \rho\}$ -central. \blacksquare

Finally, we state a proposition that will be used in Sections 11.2 and 11.3.

Proposition 11.14 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. Then, for any bonsai B_ℓ that is S_k -dominated for some $k \in \{1, 2\}$, $J_\ell^2 = \emptyset$ and the bonsai matrix N_ℓ is a network matrix.*

Proof. Let B_ℓ be some S_1 -dominated bonsai. (The case S_2 instead of S_1 is similar.) If B_ℓ is on the left of $\{e_1, e_\rho\}$, then all B_ℓ -paths are leaving $v_{\ell,1}$. By Lemma 7.7 and definition of \sim_{E_ℓ} , $J_\ell^2 = \emptyset$. Clearly, the bonsai B_ℓ is a $v_{\ell,1}$ -rooted network representation of N_ℓ . If B_ℓ is on the right of $\{e_1, e_\rho\}$, the conclusion results from Lemma 11.8. \blacksquare

11.2 The procedure Central I

Throughout this section we assume that $S_0 = \emptyset$, and $\epsilon = 1$ (except in the procedures). We provide a proof of Theorem 11.3. Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. If we look at the succession of bonsais intersecting the path from w_1 to $w_{\rho-1}$ in $G_0(A)$, we may identify a first group (maybe empty) of S_1 -dominated bonsais, a second with shared bonsais and a third with S_2 -dominated bonsais. This motivates the following definition. Given a $\{1, \rho\}$ -central representation $G(A)$ of A , a nonempty ordered set U of at most two shared bonsais is said to be *left-extreme* if the following holds.

- a) $U = (E_u)$.
The bonsai B_u intersects the basic cycle on the left of $\{e_1, e_\rho\}$ and for any shared bonsai B_ℓ intersecting the basic cycle on the left of $\{e_1, e_\rho\}$, we have $v_{\ell,1} = v_{\ell,2} \in B_u$.
- b) $U = (E_u, E_{u'})$.
The bonsais B_u and $B_{u'}$ contain each one at least one edge of the basic cycle, B_u preceding $B_{u'}$ and any shared bonsai containing edges of the basic cycle is succeeding B_u and preceding $B_{u'}$.

In the case where $|U| = 2$, U is also called a *left-extreme pair*. When it does not exist a shared bonsai containing at least one edge of the basic cycle, a left-extreme set may be not unique. In Figure 11.2, the ordered pair (E_1, E_2) is left-extreme. The notion of left-extreme set is very important and has inspired the definition of left-compatible set which will be given later. Lemma 11.17 below shows that a left-extreme set is left-compatible. The procedure

CentralI is based on a subroutine which is performed for every left-compatible set, as long as a $\{1, \rho\}$ -central representation of A has not been found. On the other hand, a U -spanning pair (j_1, j_2) is such that $j_1 \in S_1$, $j_2 \in S_2$ and if $|U| = 2$ and A has a $\{1, \rho\}$ -central representation $G(A)$, then the whole basic cycle in $G(A)$ is "spanned" by the union of the fundamental circuits of f_{j_1} and f_{j_2} . The following proposition deals with the case where there is no shared bonsai on the left of $\{e_1, e_\rho\}$ in some $\{1, \rho\}$ -central representation of A .

Proposition 11.15 *Suppose that $S_0 = \emptyset$ and there exists a $\{1, \rho\}$ -central representation $G(A)$ of A such that all bonsais on the left of $\{e_1, e_\rho\}$ are not shared. Then A is a network matrix.*

Proof. Let $G(A)$ be a basic $\{1, \rho\}$ -central representation of A such that all bonsais on the left of $\{e_1, e_\rho\}$ are not shared. So one can cut the cycle at some node, by separating the S_1 -dominated bonsais from the S_2 -dominated ones. Then, replacing the bidirected edge $e_1 = [w_1, w_\rho]$ by $]w_1, w_\rho]$ and reversing the orientation of all edges in the maximal subtree containing w_1 but not e_1 yields a network representation of A . ■

Using Proposition 11.15, since A is not a network matrix, whenever A is $\{1, \rho\}$ -central, there always exists a left-extreme set of bonsais. Before formulating the notions of left-compatible set and U -spanning pair precisely, we study the following problem. Provided that A has a $\{1, \rho\}$ -central representation and given a shared bonsai B_ℓ on the left of $\{e_1, e_\rho\}$, is it possible to compute the indexes of edges belonging to the basic cycle and B_ℓ ? Surprisingly, the answer is yes, provided that the left-extreme set of bonsais is known and of cardinality 2, and using the nonnegativity of the matrix A .

For any shared bonsai E_u ($1 \leq u \leq b$), we define

$$I_\cap(E_u) = \bigcup_{j \in S_1, j' \in S_2} E_u \cap s(A_{\bullet j}) \cap s(A_{\bullet j'})$$

and

$$I(E_u) = \bigcup_{\substack{j \in f^*(E_\ell) \cap f^*(E_u) \\ E_\ell: \text{sensitive}}} E_{uj}.$$

A vertex β in the local connector set of E_u ($\beta \in \bar{f}^*(E_u)$) is said to be S_k -blue, for some $k \in \{1, 2\}$, if $s(A_{\bullet \beta}) \cap E_{uS_k} \neq \emptyset$. For any shared bonsai E_u ($1 \leq u \leq b$), under a certain condition, the following procedure computes a subset of E_u denoted by $I_k(E_u)$, for some $k \in \{1, 2\}$, whose interpretation appears in the next lemma.

Procedure Create-Ik(E_u, k)

Input: A shared bonsai E_u and an index $k \in \{1, 2\}$.

Output: Either a row index subset $I_k(E_u)$ of E_u , or determines that A has no $\{1, \rho\}$ -central representation with a left-extreme pair in which E_u is the first bonsai for $k = 1$ (respectively, second for $k = 2$).

- 1) **if** E_u is jointly shared, **then**
 let $I_k(E_u) = \bigcap_{j \in f_{S_k}(E_u)} \{E_{uj} : I_\cap(E_u) \subseteq E_{uj}\}$;
 otherwise
- 2) compute a yellow minimal path h in $H_{E_u S^*}(A_{E_u \times \bar{f}^*(E_u)})$ from some S_1 -blue vertex β_1

- to a S_2 -blue β_2 ; let $R_k = s(A_{\bullet\beta_k}) \cap E_{uS_k}$;
- 3) compute $j_k \in S_k$ such that $R_k \subseteq E_{uj_k}$; if such a vertex (j_k) in S_k does not exist, then STOP: output that A has no $\{1, \rho\}$ -central representation with a left-extreme pair, in which E_u is the first bonsai for $k = 1$ (resp., second for $k = 2$);
 - 4) **if** the length of h is even, **then**
 - let $I_k(E_u) = \cup_{j \in f_{S_k}(E_u)} \{E_{uj} : E_{uj} \cap R_k = \emptyset; E_{uj} \subseteq E_{uj_k}\}$;
 - otherwise**
 - let $I_k(E_u) = \cap_{j \in f_{S_k}(E_u)} \{E_{uj} : R_k \subseteq E_{uj}\}$;
 - endif**
 - endif**
 - output $I_k(E_u)$;

Lemma 11.16 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$ and assumption \mathcal{A} is satisfied. Let U be a left-extreme set of bonsais. Then we have*

- (i) *If some bonsai $E_\ell \notin U$ is jointly shared and B_ℓ is on the left of $\{e_1, e_\rho\}$, then $I_\cap(E_\ell)$ corresponds to the index set of edges belonging to B_ℓ and the basic cycle.*
- (ii) *If $U = (E_u)$ and there exists at least one sensitive bonsai distinct from B_u , then $I(E_u)$ corresponds to the index set of edges belonging to B_u and the basic cycle.*
- (iii) *If $U = (E_u, E_{u'})$, then $I_1(E_u)$ (respectively, $I_2(E_{u'})$) output by the procedure Create- I_k corresponds to the index set of edges belonging to B_u (respectively, $B_{u'}$) and the basic cycle.*

Proof.

- (i) Since any edge with index in $I_\cap(E_\ell)$ lies on the basic cycle, from the definition of U , it follows that $|U| = 2$ and $I_\cap(E_\ell) \subseteq I(v_{\ell,1}, v_{\ell,2})$. Let $U = (E_u, E_{u'})$. By definition of U , B_ℓ is succeeding B_u and preceding $B_{u'}$. This implies that for any $j \in f_{S_1}(E_{u'})$ and $j' \in f_{S_2}(E_u)$, $I(v_{\ell,1}, v_{\ell,2}) = E_u \cap s(A_{\bullet j}) \cap s(A_{\bullet j'}) \subseteq I_\cap(E_\ell)$. Thus $I_\cap(E_\ell) = I(v_{\ell,1}, v_{\ell,2})$.
- (ii) Suppose that $U = (E_u)$ and there exists at least one sensitive bonsai distinct from B_u . For any sensitive bonsai B_ℓ , $\ell \neq u$, let us prove that $v_{\ell,1} = v_{\ell,2}$, and $v_{\ell,1} = v_{u,1}$ or $v_{\ell,1} = v_{u,2}$.

The fact that $v_{\ell,1} = v_{\ell,2}$ follows from the definition of U . Moreover, by Lemma 11.7, $v_{\ell,1}$, $v_{u,1}$ and $v_{u,2}$ lie on the basic cycle. If $v_{\ell,1} \notin B_u$, then we may assume that B_ℓ is succeeding B_u , hence $p(v_{\ell,1}, v_{u,2})$ is contained in the fundamental circuit of f_j and $f_{j'}$ for any $j \in f_{S_1}(E_\ell)$ and $j' \in f_{S_2}(E_u)$, which contradicts the definition of U . Thus $v_{\ell,1} \in B_u$. Suppose, to the contrary, that $v_{\ell,1}$ is an inner node of the path $p(v_{u,1}, v_{u,2})$. Then, for any $j \in f_{S_1}(E_\ell)$ and $j' \in f_{S_2}(E_\ell)$, B_{uj} (respectively, $B_{uj'}$) is a B_u -path entering (respectively, leaving) v_ℓ . So B_u is S_1, S_2 linked to B_ℓ , and by assumption \mathcal{A} , we have $J_u^2 = \emptyset$ and N_u is a network matrix. Hence, since B_u is a network representation of $A_{E_u \times \bar{f}^*(E_u)}$, Lemma 7.10 implies that $E_{uj} \approx_{E_u} E_{uj'}$, which contradicts $J_u^2 = \emptyset$. Thus, $v_{\ell,1} = v_{u,1}$ or $v_{\ell,1} = v_{u,2}$.

Therefore, either $f^*(E_\ell) \cap f^*(E_u) = \emptyset$ and $v_{u,1} = v_{u,2}$, or $f^*(E_\ell) \cap f^*(E_u) \neq \emptyset$ and $E_{uj} = I(v_{u,1}, v_{u,2})$ for all $j \in f^*(E_\ell) \cap f^*(E_u)$. This implies that $I(E_u)$ represents the index set of edges belonging to B_u and the basic cycle.

- (iii) Suppose that $U = (E_u, E_{u'})$. We prove that $I_1(E_u) = I(v_{u,1}, v_{u,2})$. (By symmetry, it can be proved that $I_2(E_{u'}) = I(v_{u',1}, v_{u',2})$.) By definition of a left-extreme pair, B_u contains at least one edge of the basic cycle and is shared. Therefore, by Lemma 11.7, $v_{u,1} \neq v_{u,2}$ and the path $p(v_{u,1}, v_{u,2})$ corresponds to the intersection of B_u with the basic cycle in $G(A)$. Notice that $f_{S_1}(E_{u'}) \neq \emptyset$ and $p(v_{u,1}, v_{u,2})$ is in the fundamental circuit of any edge with index in $f_{S_1}(E_{u'})$. Thus

$$I(v_{u,1}, v_{u,2}) = E_{uj} \text{ for all } j \in f_{S_1}(E_{u'}). \quad (11.1)$$

Moreover, recall that $T_{u,k}$ denotes the tree with edge index set E_{uS_k} for $k = 1$ and 2 . By Lemma 11.6 (i) and statement (11.1), it results that

$$T_{u,1} \cap T_{u,2} \text{ is a subpath of } p(v_{u,1}, v_{u,2}) \text{ with one endnode equal to } v_{u,2}. \quad (11.2)$$

Suppose first that B_u is jointly shared. So $I_\cap(E_u) \neq \emptyset$. Let $j \in f_{S_1}(E_u)$ such that $I_\cap(E_u) \subseteq E_{uj}$. Since $I_\cap(E_u)$ is the edge index set of the subgraph $T_{u,1} \cap T_{u,2}$, by (11.2), it follows that the node $v_{u,2}$ is contained in the B_u -path B_{uj} , and clearly $v_{u,1} \in B_{uj}$; hence $I(v_{u,1}, v_{u,2}) \subseteq E_{uj}$. So $I(v_{u,1}, v_{u,2}) \subseteq I_1(E_u)$. Moreover, using (11.1), we have that

$$\begin{aligned} I_1(E_u) &= \bigcap_{j \in f_{S_1}(E_u)} \{E_{uj} : I_\cap(E_u) \subseteq E_{uj}\} \\ &\subseteq \bigcap_{j \in f_{S_1}(E_{u'})} \{E_{uj} : I_\cap(E_u) \subseteq E_{uj}\} \\ &= I(v_{u,1}, v_{u,2}). \end{aligned}$$

We conclude that $I(v_{u,1}, v_{u,2}) = I_1(E_u)$. See Figure 11.6 for an example.

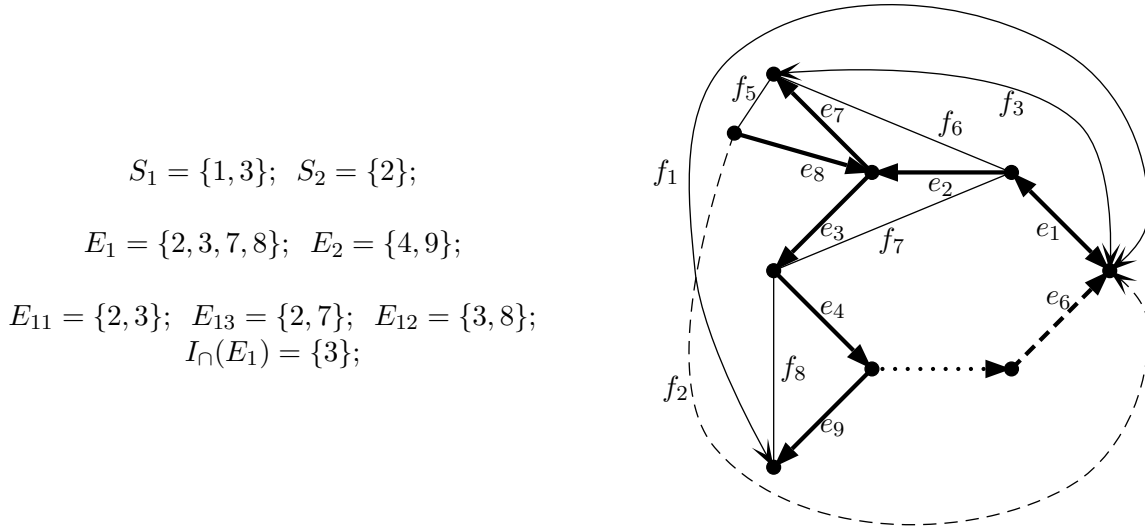


Figure 11.6: A (partial) $\{1, 6\}$ -central representation of some binet $\{0, 1\}$ -matrix, where the pair (E_1, E_2) is left-extreme and E_1 is jointly shared.

Now suppose that B_u is disjointly shared. Let h be a yellow minimal path in $H_{E_u S^*}(A_{E_u \times \bar{f}^*(E_u)})$ from some S_1 -blue vertex β_1 to a S_2 -blue β_2 ; let $R_i = s(A_{\bullet\beta_i}) \cap E_{uS_i}$

for $i = 1$ and 2 . Denote by p_1 and p_2 the paths with edge index sets R_1 and R_2 , respectively. Since A is nonnegative, by Lemma 4.2 p_1 is a directed path. As p_1 is contained in the tree $T_{u,1}$ consisting of directed B_u -paths leaving the node $v_{u,1}$, p_1 is contained in a B_u -path. Thus there exists $j_1 \in S_1$ such that $R_1 \subseteq E_{uj_1}$ and step 3 in the procedure Create-Ik is justified. Then, as B_u is disjointly shared, by statement (11.1), $T_{u,1} \cap T_{u,2} = \{v_{u,2}\}$. Hence, since $p_1 \subseteq T_{u,1}$, $p_2 \subseteq T_{u,2}$ and h is a yellow path in $H_{E_u S^*}(A_{E_u \times \bar{f}^*(E_u)})$, one can prove that $v_{u,2}$ is a common endnode of the paths p_1 and p_2 . Since p_2 enters $v_{u,2}$, by Lemma 7.5 the path h has an even length if and only if p_1 leaves $v_{u,2}$.

Suppose that h has an even length. So p_1 leaves $v_{u,2}$. Since $R_1 \subseteq E_{uj_1}$, it follows that $I(v_{u,1}, v_{u,2}) \cap R_1 = \emptyset$ and $I(v_{u,1}, v_{u,2}) \subseteq E_{uj_1}$. Using (11.1), this implies that

$$\begin{aligned} I(v_{u,1}, v_{u,2}) &= \cup_{j \in f_{S_1}(E_u)} \{E_{uj} : E_{uj} \cap R_1 = \emptyset; E_{uj} \subseteq E_{uj_1}\} \\ &\subseteq I_1(E_u). \end{aligned}$$

Now, if $j \in f_{S_1}(E_u)$ is such that $E_{uj} \cap R_1 = \emptyset$ and $E_{uj} \subseteq E_{uj_1}$, then $E_{uj} \subseteq I(v_{u,1}, v_{u,2})$. Therefore $I_1(E_u) = I(v_{u,1}, v_{u,2})$.

Finally, suppose that h has an odd length. So p_1 is entering $v_{u,2}$ and has to be contained in $p(v_{u,1}, v_{u,2})$. So $R_1 \subseteq I(v_{u,1}, v_{u,2})$, and using (11.1) this implies that

$$\begin{aligned} I_1(E_u) &\subseteq \cap_{j \in f_{S_1}(E_u)} \{E_{uj} : R_1 \subseteq E_{uj}\} \\ &= I(v_{u,1}, v_{u,2}). \end{aligned}$$

Now, if $j \in f_{S_1}(E_u)$ is such that $R_1 \subseteq E_{uj}$, then $I(v_{u,1}, v_{u,2}) \subseteq E_{uj}$. Thus $I_1(E_u) = I(v_{u,1}, v_{u,2})$. This concludes the proof. ■

For all $1 \leq \ell \leq b$, let $E'_\ell = E_\ell \cup \{1, \rho\}$,

$$A_\ell = A_{E'_\ell \times f(E_\ell)} \text{ and } A_\ell^\cap = [A_\ell \chi_{I \cap (E_\ell) \cup \{1, \rho\}}^{E'_\ell}].$$

The row indexes 1 and ρ of a matrix A_ℓ ($1 \leq \ell \leq b$) are called *artificial* as well as the corresponding edges in any network representation of A_ℓ (if one exists). For any shared bonsai E_u ($1 \leq u \leq b$), let

$$L_u = [A_u \chi_{I(E_u) \cup \{1, \rho\}}^{E'_u}]$$

if there exists a sensitive bonsai distinct from E_u , and $L_u = A_u$ otherwise. Suppose that for some $k \in \{1, 2\}$, the procedure Create-Ik with input E_u and k has output a nonempty subset $I_k(E_u)$ of E_u . Then we define

$$L_{u,k} = [A_u \chi_{I_k(E_u) \cup \{1, \rho\}}^{E'_u}].$$

A nonempty ordered set U of at most two shared bonsais is said to be *left-compatible* if the following holds.

case: $U = (E_u)$.

The matrix L_u has a network representation in which e_1 and e_ρ are nonalternating.

case: $U = (E_u, E_{u'})$.

The procedure Create-Ik with input E_u and index 1 (resp., $E_{u'}$ and index 2) outputs a nonempty set $I_1(E_u)$ (resp., $I_2(E_{u'})$). Moreover, $L_{u,1}$ and $L_{u',2}$ are network matrices, $f_{S_1}(E_{u'}) \subseteq \{\beta \in f_{S_1}(E_u) : E_{u\beta} = I_1(E_u)\}$ and $f_{S_2}(E_u) \subseteq \{\beta \in f_{S_2}(E_{u'}) : E_{u'\beta} = I_2(E_{u'})\}$.

$$E_1 = \{3, 4, 7, 8, 9, 10, 11\};$$

	f_3	f_4	f_5	f_6	f_7
e_7	1	1	0	0	0
e_8	0	1	1	0	0
e_9	0	0	1	1	0
e_{10}	0	0	0	1	1
e_{11}	1	0	0	0	1

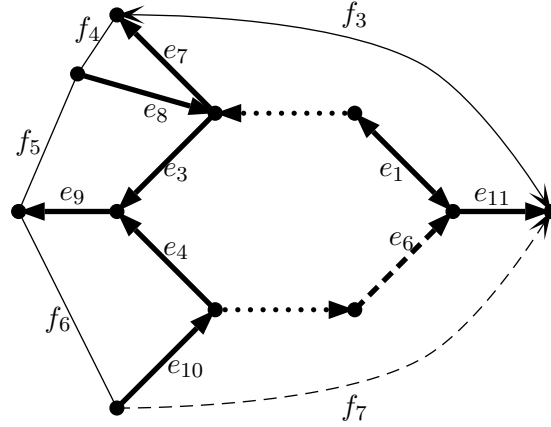


Figure 11.7: A $\{1, 6\}$ -central representation of some binet $\{0, 1\}$ -matrix A on the right, where (E_1) is left-compatible and left-extreme, and a non-network submatrix of A on the left.

$$L_1 =$$

	f_3	f_4	f_5	f_6	f_7
e_3	0	0	1	0	0
e_4	0	0	0	1	0
e_7	1	1	0	0	0
e_8	0	1	1	0	0
e_9	0	0	1	1	0
e_{10}	0	0	0	1	1
e_1	1	0	0	0	0
e_6	0	0	0	0	1

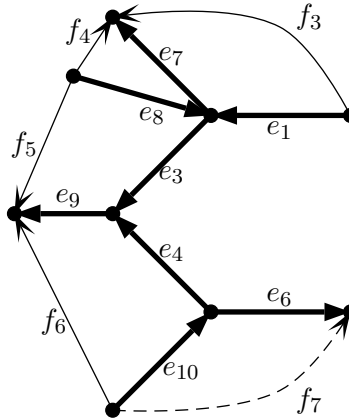


Figure 11.8: The matrix $L_1 = A_1$ and a network representation of it such that e_1 and e_6 are nonalternating, with respect to the binet matrix whose a $\{1, 6\}$ -central representation is given in Figure 11.7.

The definition of left-compatible set is motivated by the next lemma. For an illustration, see Figures 11.7 and 11.8.

Lemma 11.17 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. Let U be a left-extreme set of bonsais. Then U is left-compatible.*

Proof. Suppose first that $U = (E_u)$. By definition, B_u is shared. If there is no sensitive bonsai distinct from B_u , then $I(E_u) = \emptyset$, otherwise $I(E_u) = I(v_{u,1}, v_{u,2})$ by Lemma 11.16 (ii). Consider the subgraph B_u of $G(A)$ with one more basic edge e_1 entering $v_{u,1}$ and a basic edge e_ρ leaving $v_{u,2}$. This yields a basic network representation of L_u in which e_1 and e_ρ are nonalternating. The proof in the case $U = (E_u, E_{u'})$ is similar using Lemma 11.16 (iii). ■

Given a left-compatible set U , we define a U -spanning pair (j_1, j_2) of column indexes as follows. If $U = (E_u)$, then $j_1 \in f_{S_1}(E_u)$, $j_2 \in f_{S_2}(E_u)$. If $U = (E_u, E_{u'})$, then $j_1 \in f_{S_1}(E_{u'})$, $j_2 \in f_{S_2}(E_u)$. Observe that such a pair always exists. For any U -spanning pair (j_1, j_2) , let

$$V(j_1, j_2) = \{E_\ell \in V : j_1 \text{ or } j_2 \in f^*(E_\ell) \text{ or } E_\ell \text{ is shared}\}$$

$$\text{and } R(j_1, j_2) = \cup_{E_\ell \in V(j_1, j_2)} E_\ell \cup \{1, \rho\}.$$

Lemma 11.18 *Let U be a left-compatible set and (j_1, j_2) a U -spanning pair. If A has a $\{1, \rho\}$ -central representation $G(A)$ such that U is left-extreme, then $R(j_1, j_2)$ is the edge index set of a basic 1-tree.*

Proof. Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$ such that U is left-extreme. Then, by definition of a U -spanning pair, it follows that $s(A_{\bullet j_1}) \cup s(A_{\bullet j_2})$ is the edge index set of a basic 1-tree. Let B_ℓ be a shared bonsai. If B_ℓ is on the left of $\{e_1, e_\rho\}$, then by Lemma 11.7 $v_{\ell,1}$ and $v_{\ell,2}$ are on the basic cycle. If B_ℓ is on the right of $\{e_1, e_\rho\}$, then any bonsai $B_{\ell'}$ containing an edge of the basic path from v_ℓ to w_ρ has to be shared. This completes the proof. ■

Given a left-compatible set U and a U -spanning pair (j_1, j_2) , the Proposition 11.19 below informs us on the bipartiteness of the graph $F^*(\overline{V(j_1, j_2)})$, in case where A is a $\{1, \rho\}$ -central matrix. By assuming that $F^*(\overline{V(j_1, j_2)})$ is bipartite, we denote by $\mathcal{U}_1, \dots, \mathcal{U}_\xi$ the connected components of $F^*(\overline{V(j_1, j_2)})$ and for each connected component \mathcal{U}_κ ($1 \leq \kappa \leq \xi$), let $\mathcal{U}_\kappa = \mathcal{U}_\kappa^0 \uplus \mathcal{U}_\kappa^1$ be a bipartition of \mathcal{U}_κ into two colour classes. For any $E_\ell \in \overline{V(j_1, j_2)}$, let

$$\text{Opposite}(E_\ell) = (\cup_{j \in f^*(E_\ell)} s(A_{\bullet j}) - \cap_{j \in f^*(E_\ell)} s(A_{\bullet j})) \cap R(j_1, j_2)$$

, and for any $1 \leq \kappa \leq \xi$ and $i \in \{0, 1\}$, let $\text{Opposite}(\mathcal{U}_\kappa^i) = \cup_{E_\ell \in \mathcal{U}_\kappa^i} \text{Opposite}(E_\ell)$. The following propositions show a part of Theorem 11.3.

Proposition 11.19 *Let U be a left-compatible set and (j_1, j_2) a U -spanning pair. Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$ such that U is left-extreme. Then*

- (i) *The graph $F^*(\overline{V(j_1, j_2)})$ is bipartite.*
- (ii) *For any $1 \leq \kappa \leq \xi$, $i, i' \in \{0, 1\}$, $E_\ell \in \mathcal{U}_\kappa^i$ and $E_{\ell'} \in \mathcal{U}_\kappa^{i'}$, B_ℓ and $B_{\ell'}$ are at different sides of $\{e_1, e_\rho\}$ if and only if $i \neq i'$.*
- (iii) *For any $1 \leq \kappa \leq \xi$, $i \in \{0, 1\}$ and $E_\ell \in \mathcal{U}_\kappa^i$, the bonsai B_ℓ and the basic subgraph of $G(A)$ with edge index set $\text{Opposite}(\mathcal{U}_\kappa^i)$ are on both sides of $\{e_1, e_\rho\}$.*

Proof. Let E_ℓ and $E_{\ell'}$ be two bonsais in $\overline{V(j_1, j_2)}$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$. Let $j \in f^*(E_\ell) \cap f^*(E_{\ell'})$. Assume $j \in S_1$ (the case $j \in S_2$ is similar). Since $V(j_1, j_2)$ contains all shared bonsais, B_ℓ and $B_{\ell'}$ are S_1 -dominated and they both contain at least one edge of the fundamental circuit of f_j . Following the lines of the proof of Proposition 7.15, one can prove that if B_ℓ and $B_{\ell'}$ are on the same side of $\{e_1, e_\rho\}$, then $(E_\ell, E_{\ell'})_{E_{\ell'j}} \in D$ or $(E_{\ell'}, E_\ell)_{E_{\ell j}} \in D$. Thus if E_ℓ and $E_{\ell'}$ are adjacent in $F^*(\overline{V(j_1, j_2)})$, then B_ℓ and $B_{\ell'}$ are at different sides of $\{e_1, e_\rho\}$. This induces a partition of $F^*(\overline{V(j_1, j_2)})$ into two colour classes, one for the bonsais on the left of $\{e_1, e_\rho\}$ and the other for those on the right of $\{e_1, e_\rho\}$. This completes the proof of (i) and (ii). ■

Now let $1 \leq \kappa \leq \xi$, $i \in \{0, 1\}$ and $E_\ell \in \mathcal{U}_\kappa^i$. We assume that B_ℓ is S_1 -dominated (the case S_2 -dominated is similar). If B_ℓ is on the left of $\{e_1, e_\rho\}$, then the path from $v_{\ell,1}$ to w_1 in $G_0(A)$ is in the fundamental circuit of every nonbasic edge with index in $f^*(E_\ell)$. Since B_ℓ does not contain any edge of the basic 1-tree with edge index set $R(j_1, j_2)$ (see Lemma 11.18), we deduce that the basic subgraph of $G(A)$ with edge index set $Opposite(E_\ell)$ is on the right of $\{e_1, e_\rho\}$. Similarly, if B_ℓ is on the right of $\{e_1, e_\rho\}$, then the basic subgraph of $G(A)$ with edge index set $Opposite(E_\ell)$ is on the left of $\{e_1, e_\rho\}$. Combining this and part (ii) gives the desired result. ■

Let U be a left-compatible set and (j_1, j_2) a U -spanning pair. We define relations \prec^{j_1} and \prec^{j_2} on $V(j_1, j_2)$. For $k = 1$ and 2 and $E_\ell, E_{\ell'} \in V(j_1, j_2)$,

$$E_\ell \prec^{j_k} E_{\ell'} \Leftrightarrow f_{S_k}(E_{\ell'}) \subseteq \{\beta \in f_{S_k}(E_\ell) : E_{\ell\beta} = E_{\ell j_k}\}.$$

Clearly, the relations \prec^{j_1} and \prec^{j_2} are transitive. One can prove that if $U = (E_u, E_{u'})$, then $E_u \prec^{j_1} E_{u'}$ and $E_{u'} \prec^{j_2} E_u$ (see the proof of Claim 2 at page 156). A bonsai $E_\ell \in V(j_1, j_2) \setminus U$ is said to be *right-feasible* if

$\omega.0$ $J_\ell^2 = \emptyset$ and N_ℓ is a network matrix;

and *left-feasible* if the following holds.

- If E_ℓ is disjointly shared, then

$\omega.1$ E_ℓ is sensitive;

$\omega.2$ for all $E_u \in U$, E_u is not S_1, S_2 linked to E_ℓ ;

$\omega.3$ if $U = (E_u)$, then $E_{uj} = I(E_u)$ for all $j \in f_{S_k}(E_\ell)$ and some $k \in \{1, 2\}$;

$\omega.4$ If $U = (E_u, E_{u'})$, then either $E_{uj} = I_1(E_u) = I_2(E_{u'})$ for all $j \in f_{S_k}(E_\ell)$ and some $k \in \{1, 2\}$, or $E_{uj} = I_1(E_u)$ and $E_{u'j'} = I_2(E_{u'})$ for all $j \in f_{S_1}(E_\ell)$ and $j' \in f_{S_2}(E_\ell)$;

- If E_ℓ is jointly shared, then

$\omega.5$ A_ℓ^\cap is a network matrix;

$\omega.6$ $|U| = 2$ and writing $U = (E_u, E_{u'})$, we have $I_\cap(E_\ell) = E_{\ell j_1} = E_{\ell j_2}$, $E_u \prec^{j_1} E_\ell \prec^{j_1} E_{u'}$ and $E_{u'} \prec^{j_2} E_\ell \prec^{j_2} E_u$.

- If E_ℓ is S_k -dominated for some $k \in \{1, 2\}$, then

- $\omega.7$ $J_\ell^2 = \emptyset$ and N_ℓ is a network matrix;
 $\omega.8$ $E_u \prec^{j_k} E_\ell$ for all $E_u \in U \cup \{E_{\ell'} : E_{\ell'} \text{ is sensitive ; } j_k \in f^*(E_{\ell'})\}$ or $E_\ell \prec^{j_k} E_u$ for all $E_u \in U \cup \{E_{\ell'} : E_{\ell'} \text{ is sensitive}\}$.

Lemma 11.20 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$ and assumption \mathcal{A} is satisfied. Let U be a left-extreme set of bonsais, (j_1, j_2) a U -spanning pair and $E_\ell \in V(j_1, j_2) \setminus U$. If B_ℓ is on the right of $\{e_1, e_\rho\}$, then E_ℓ is right-feasible, otherwise left-feasible.*

Proof. If B_ℓ is on the right of $\{e_1, e_\rho\}$, then by Lemma 11.8 it is right-feasible. Now assume that B_ℓ is on the left of $\{e_1, e_\rho\}$.

Suppose first that B_ℓ is disjointly shared. Clearly, B_ℓ does not contain any edge of the basic cycle (otherwise $U = (E_u, E_{u'})$ for some $1 \leq u, u' \leq b$ and B_ℓ would be succeeding B_u and preceding $B_{u'}$, which implies that $I_\cap(E_\ell) \neq \emptyset$). By Lemma 11.7, $v_{\ell,1} = v_{\ell,2}$ and B_ℓ is sensitive. For any $E_u \in U$, by Lemma 11.12, B_u is not S_1, S_2 linked to B_ℓ ; this implies that B_ℓ is either preceding or succeeding B_u . In case where $U = (E_u)$, using Lemma 11.16 (ii), it follows that $E_{uj} = I(E_u)$ for all $j \in f_{S_1}(E_\ell)$ if E_ℓ is succeeding E_u , or $E_{uj} = I(E_u)$ for all $j \in f_{S_2}(E_\ell)$ otherwise. In a similar way using Lemma 11.16 (iii), one can deal with the case $|U| = 2$, proving that E_ℓ is left-feasible.

Secondly, suppose that E_ℓ is jointly shared. By Lemma 11.16 (i), B_ℓ contains at least one edge of the basic cycle. So, by definition of U , $U = (E_u, E_{u'})$ for some $1 \leq u, u' \leq b$, and B_ℓ is succeeding B_u and preceding $B_{u'}$. Then, using Lemma 11.16 (i) and since $j_1 \in f_{S_1}(E_{u'})$ and $j_2 \in f_{S_2}(E_u)$, we have $I_\cap(E_\ell) = E_{\ell j_1} = E_{\ell j_2}$, $E_u \prec^{j_1} E_\ell \prec^{j_1} E_{u'}$ and $E_{u'} \prec^{j_2} E_\ell \prec^{j_2} E_u$. Thus E_ℓ is left-feasible.

Finally, suppose that E_ℓ is S_1 -dominated (the case S_2 instead of S_1 is symmetric). By Proposition 11.14, $J_\ell^2 = \emptyset$ and N_ℓ is a network matrix. Moreover, since $E_\ell \in V(j_1, j_2)$, $j_1 \in f^*(E_\ell)$. Then, either B_ℓ contains at least one edge of the basic cycle and so $E_\ell \prec^{j_1} E_u$ for all $E_u \in U \cup \{E_{\ell'} : E_{\ell'} \text{ is sensitive}\}$, or $E_u \prec^{j_1} E_\ell$ for all $E_u \in U \cup \{E_{\ell'} : E_{\ell'} \text{ is sensitive ; } j_1 \in f^*(E_{\ell'})\}$. This completes the proof. \blacksquare

Let U be a left-compatible set and (j_1, j_2) a U -spanning pair. A pair of bonsais $E_\ell, E_{\ell'} \in V(j_1, j_2) - U$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$ is said to be *right-feasible* if

$$\phi.1 \quad (E_\ell, E_{\ell'}) \in D \text{ or } (E_{\ell'}, E_\ell) \in D;$$

and *left-feasible* if the following holds.

$$\phi.2 \quad \text{The bonsais } E_\ell \text{ and } E_{\ell'} \text{ are not both sensitive.}$$

$$\phi.3 \quad \text{If } E_\ell \text{ and } E_{\ell'} \text{ are jointly shared, then } E_\ell \prec^{j_k} E_{\ell'} \text{ and } E_{\ell'} \prec^{j_{k'}} E_\ell \text{ for some } k, k' \in \{1, 2\}, k \neq k'.$$

$$\phi.4 \quad \text{If } E_\ell \text{ and } E_{\ell'} \text{ are } S_k\text{-dominated for some } k \in \{1, 2\}, \text{ then } E_\ell \prec^{j_k} E_{\ell'} \text{ or } E_{\ell'} \prec^{j_k} E_\ell.$$

$$\phi.5 \quad \text{If } E_\ell \text{ is jointly shared and } E_{\ell'} \text{ sensitive, then } E_\ell \text{ is not } S_1, S_2 \text{ linked to } E_{\ell'} \text{ and } E_\ell \prec^{j_k} E_{\ell'} \text{ for some } k \in \{1, 2\}.$$

Lemma 11.21 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$ and assumption \mathcal{A} is satisfied. Let U be a left-extreme set of bonsais and (j_1, j_2) a U -spanning pair. Let $E_\ell, E_{\ell'} \in V(j_1, j_2) - U$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$. If B_ℓ and $B_{\ell'}$ are both on the right (respectively, the left) of $\{e_1, e_\rho\}$, then the pair $E_\ell, E_{\ell'}$ is right-feasible (respectively, left-feasible).*

Proof. Let us first remark that since U is left-extreme, by Lemma 11.17, U is left-compatible. So (j_1, j_2) is well defined. Let $E_\ell, E_{\ell'} \in V(j_1, j_2) \setminus U$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$. If E_ℓ and $E_{\ell'}$ are both on the right of $\{e_1, e_\rho\}$, then using Proposition 7.14, the pair is right-feasible.

Suppose now that B_ℓ and $B_{\ell'}$ are on the left of $\{e_1, e_\rho\}$. The bonsais E_ℓ and $E_{\ell'}$ can not be both sensitive (otherwise from Lemmas 11.9 and 11.12, it follows that $v_{\ell,1} = v_{\ell,2}$ and $v_{\ell',1} = v_{\ell',2}$, hence $f^*(E_\ell) \cap f^*(E_{\ell'}) = \emptyset$). If E_ℓ and $E_{\ell'}$ are jointly shared, by Lemma 11.20, we have that $I_\cap(E_\ell) = E_{\ell j_1} = E_{\ell j_2}$ and $I_\cap(E_{\ell'}) = E_{\ell' j_1} = E_{\ell' j_2}$. Since B_ℓ is preceding $B_{\ell'}$ or conversely, it follows that the pair $E_\ell, E_{\ell'}$ is left-feasible. The proof is also straightforward, using Proposition 7.14, in case where E_ℓ and $E_{\ell'}$ are both S_k -dominated for some $k \in \{1, 2\}$.

At last, suppose that B_ℓ is jointly shared and $B_{\ell'}$ is sensitive. By Lemma 11.20 and 11.16 (i), $I_\cap(E_\ell) = E_{\ell j_1} = E_{\ell j_2}$ is the index set of edges in B_ℓ and the basic cycle. Then, by Lemma 11.12, $B_{\ell'}$ is not S_1, S_2 linked to B_ℓ , so that it is either preceding B_ℓ in which case $E_\ell \prec^{j_2} E_{\ell'}$, or succeeding B_ℓ and $E_\ell \prec^{j_1} E_{\ell'}$. ■

Assume that the graph $F^*(\overline{V(j_1, j_2)})$ is bipartite. Given a left-compatible set U and a U -spanning pair (j_1, j_2) , we are now ready to construct the instance $\Omega(U, j_1, j_2)$ of the 2-SAT problem. The set of variables is $\{x_\ell : E_\ell \in V(j_1, j_2)\}$. Provided that A is $\{1, \rho\}$ -central, the equalities $x_\ell = 0$ and $x_\ell = 1$ mean that the bonsai B_ℓ is on the left and on the right of $\{e_1, e_\rho\}$, respectively, in some $\{1, \rho\}$ -central representation of A .

For any $E_u \in U$, set $x_u = 0$ in $\Omega(U, j_1, j_2)$. For any $E_\ell \in V(j_1, j_2) \setminus U$, if E_ℓ is not right-feasible (respectively, left-feasible), then let $x_\ell = 0$ (respectively, $x_\ell = 1$). For any pair of bonsais $E_\ell, E_{\ell'} \in V(j_1, j_2) - U$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$, if the pair is not right-feasible, then put the clause $\bar{x}_\ell \vee \bar{x}_{\ell'}$ in $\Omega(U, j_1, j_2)$. Furthermore, if the pair is not left-feasible, then put the clause $x_\ell \vee x_{\ell'}$ in $\Omega(U, j_1, j_2)$. Thus if the pair is not right-feasible (respectively, left-feasible), then at most one of the variables x_ℓ and $x_{\ell'}$ has value 1 (respectively, 0).

For $\kappa = 1, \dots, \xi$, do as follows. For any two variables x_ℓ and $x_{\ell'}$ such that $E_\ell \cap \text{Opposite}(\mathcal{U}_\kappa^i) \neq \emptyset$ and $E_{\ell'} \cap \text{Opposite}(\mathcal{U}_\kappa^i) \neq \emptyset$ with $i = 0$ or 1 , put the equality $x_\ell = x_{\ell'}$ in $\Omega(U, j_1, j_2)$. Moreover, if $\text{Opposite}(\mathcal{U}_\kappa^0) \neq \emptyset$ and $\text{Opposite}(\mathcal{U}_\kappa^1) \neq \emptyset$, choose some x_ℓ and $x_{\ell'}$ such that $E_\ell \cap \text{Opposite}(\mathcal{U}_\kappa^0) \neq \emptyset$ and $E_{\ell'} \cap \text{Opposite}(\mathcal{U}_\kappa^1) \neq \emptyset$. Then put the clauses $x_\ell \vee x_{\ell'}$ and $\bar{x}_\ell \vee \bar{x}_{\ell'}$ in $\Omega(U, j_1, j_2)$. These clauses ensure that the variables x_ℓ and $x_{\ell'}$ have different values.

Proposition 11.22 *Suppose that A has a $\{1, \rho\}$ -central representation and assumption \mathcal{A} is satisfied. Let U be a left-extreme set. Then, for any U -spanning pair (j_1, j_2) , the instance $\Omega(U, j_1, j_2)$ has a truth assignment.*

Proof. Let (j_1, j_2) be a U -spanning pair. For each bonsai E_ℓ , let $x_\ell = 0$ if E_ℓ is on the left of $\{e_1, e_\rho\}$, and $x_\ell = 1$ otherwise. By Lemmas 11.20 and 11.21 and Proposition 11.19 (ii), this is a truth assignment. ■

The following procedure computes a truth assignment of $\Omega(U, j_1, j_2)$ for some left-compatible set U and a U -spanning pair (j_1, j_2) , whenever A is $\{\epsilon, \rho\}$ -central.

Procedure Truthassignment($A, \{\epsilon, \rho\}$)

Input: A non-network matrix A and two row indexes ϵ and $\rho \neq 1$ such that $S_0 = \emptyset$.

Output: Either a left-compatible set U and a U -spanning pair (j_1, j_2) with a truth assignment of $\Omega(U, j_1, j_2)$, or determines that A is not $\{\epsilon, \rho\}$ -central.

- 1) call **Initialization** $(A, \{\epsilon, \rho\})$ outputting a matrix A' ;
- 2) compute a digraph D with respect to A' and the row index subset $R^* = \{1, \rho\}$ of A' ;
- 3) if $g_\beta(E_\ell) = 2$ for some $E_\ell \in V$, $\beta \in S^*$, then **STOP**: output that A is not $\{\epsilon, \rho\}$ -central;
- 4) **for** every ordered pair U of shared bonsais, $|U| \leq 2$, **do**
 check whether U is left-compatible; if it is not, then return to 4;
 compute a U -spanning pair (j_1, j_2) , and test if $F^*(\overline{V(j_1, j_2)})$ is bipartite;
 if it is not, then return to 4;
 compute the instance $\Omega(U, j_1, j_2)$; compute a truth assignment of $\Omega(U, j_1, j_2)$,
 if one exists, output it with U and (j_1, j_2) and **STOP**, otherwise return to 4;
endfor
 output that A is not $\{\epsilon, \rho\}$ -central;

Proposition 11.23 *The output of the procedure Truthassignment is correct.*

Proof. By Lemma 11.13, the subroutine Initialization outputs a matrix A' of size $n \times m'$ whose $A'_{\bullet\{1, \dots, m\}} = A$ and such that A' is $\{1, \rho\}$ -central if and only if A is $\{\epsilon, \rho\}$ -central, and assumption \mathcal{A} is satisfied for A' . Step 3 is justified by Lemma 11.5.

Suppose that A is $\{\epsilon, \rho\}$ -central. Then, let $G(A')$ be a $\{1, \rho\}$ -central representation of A' , and U a left-extreme set of bonsais. Then, by Lemma 11.17, U is left-compatible. Let (j_1, j_2) be a U -spanning pair. By Proposition 11.19 (i), $F^*(\overline{V(j_1, j_2)})$ is bipartite. Finally, using Proposition 11.22, the procedure Truthassignment outputs a truth assignment of $\Omega(U, j_1, j_2)$. This concludes the proof. \blacksquare

Before proving Theorem 11.3, we describe some advantages and properties which ensue from a truth assignment of $\Omega(U, j_1, j_2)$. Let U be a left-compatible set and (j_1, j_2) a U -spanning pair. If $|U| = 2$, then we write $U = (E_u, E_{u'})$, otherwise $U = (E_u)$. Suppose that each variable x_ℓ ($E_\ell \in V(j_1, j_2)$) has received a value 0 or 1, yielding a truth assignment of $\Omega(U, j_1, j_2)$. Let

$$X_0(j_1, j_2) = \{E_\ell \in V(j_1, j_2) : x_\ell = 0\}$$

and

$$X_1(j_1, j_2) = \{E_\ell \in V(j_1, j_2) : x_\ell = 1\}.$$

By $\omega.6$, notice that if E_ℓ is jointly shared and $x_\ell = 0$, then $|U| = 2$. So, up to a renumbering of the bonsais and a change of the value of u and u' , we may assume that $\{E_\ell : x_\ell = 0; j_1, j_2 \in f^*(E_\ell)\} = \{E_u, E_{u+1}, \dots, E_{u'}\}$, $\{E_\ell : x_\ell = 0; j_1 \in f^*(E_\ell)\} = \{E_1, E_2, \dots, E_{\ell_1}\}$ and $\{E_\ell : x_\ell = 0; j_2 \in f^*(E_\ell)\} = \{E_u, E_{u+1}, \dots, E_{u'}\} \cup \{E_{\ell_1+1}, E_{\ell_1+2}, \dots, E_{\ell_2}\}$, where $1 \leq u \leq u' \leq \ell_1 \leq \ell_2$ and $u = u'$ in case $|U| = 1$. Now we state some useful claims.

Claim 0. For any sensitive bonsai E_ℓ , $x_\ell = 0$ and there is no bonsai $E_{\ell'}$ ($\ell' \neq \ell$) that is S_1, S_2 linked to E_ℓ and such that $x_{\ell'} = 0$.

Proof of Claim 0. Since $J_\ell^2 \neq \emptyset$, E_ℓ is not right-feasible, so $x_\ell = 0$. Let $E_{\ell'}$ be a bonsai S_1, S_2 linked to E_ℓ . Suppose, to the contrary, that $x_{\ell'} = 0$. Thus, $E_{\ell'}$ is left-feasible, and the

pair $E_\ell, E_{\ell'}$ is left-feasible. By $\phi.5$, $E_{\ell'}$ is not jointly shared, so $E_{\ell'}$ is disjointly shared and by $\omega.2$, $E_{\ell'}$ is sensitive, contradicting $\phi.2$. \blacksquare

Claim 1. For any $E_\ell \in V(j_1, j_2)$ such that $j_1, j_2 \notin f^*(E_\ell)$, $x_\ell = 0$ if and only if E_ℓ is sensitive.

Proof of Claim 1. Let $E_\ell \in V(j_1, j_2)$ such that $j_1, j_2 \notin f^*(E_\ell)$. By definition of $V(j_1, j_2)$, it results that E_ℓ is shared. If E_ℓ is sensitive, then by Claim 0 $x_\ell = 0$. Suppose now that $x_\ell = 0$. If E_ℓ is jointly shared, then, as $I_\cap(E_\ell) \neq \emptyset$, $\omega.6$ implies that $j_1, j_2 \in f^*(E_\ell)$, a contradiction. Thus E_ℓ is disjointly shared and by $\omega.1$, E_ℓ is sensitive. \blacksquare

Claim 2. Up to a renumbering of the bonsais and a change of the value of u and u' , there exist orderings, say

$$E_1 \prec^{j_1} E_2 \cdots \prec^{j_1} E_u \prec^{j_1} E_{u+1} \cdots \prec^{j_1} E_{u'} \prec^{j_1} E_{u'+1} \cdots \prec^{j_1} E_{\ell_1} \quad (11.3)$$

and

$$E_{\ell_1+1} \prec^{j_2} E_{\ell_1+2} \cdots \prec^{j_2} E_{\ell'_2} \prec^{j_2} E_{u'} \prec^{j_2} E_{u'-1} \cdots \prec^{j_2} E_u \prec^{j_2} E_{\ell'_2+1} \prec^{j_2} E_{\ell'_2+2} \cdots \prec^{j_2} E_{\ell_2} \quad (11.4)$$

of the sets $\{E_\ell : x_\ell = 0; j_1 \in f^*(E_\ell)\}$ and $\{E_\ell : x_\ell = 0; j_2 \in f^*(E_\ell)\}$, respectively, where $l_1 \leq l'_2 \leq l_2$, $E_{u-1} \prec^{j_1} E_\ell$ (if $u \neq 1$) and $E_{\ell'_2} \prec^{j_2} E_\ell$ (if $l'_2 \neq l_1$) for any sensitive bonsai E_ℓ .

Proof of Claim 2. Let us prove first that if $U = (E_u, E_{u'})$, then $E_u \prec^{j_1} E_{u'}$ (symmetrically, we can prove that $E_{u'} \prec^{j_2} E_u$). Let $j \in f_{S_1}(E_{u'})$. By definition of the U -spanning pair (j_1, j_2) , $j_1 \in f_{S_1}(E_u)$. Moreover, by definition of the left-compatible set U , $E_{uj} = I_1(E_u)$ and $E_{uj_1} = I_1(E_u)$. Thus, $E_{uj} = E_{uj_1}$. Therefore, $f_{S_1}(E_{u'}) \subseteq \{\beta \in f_{S_1}(E_u) : E_{u\beta} = E_{uj_1}\}$.

Let E_ℓ be a bonsai not in U such that $j_1, j_2 \in f^*(E_\ell)$ and $x_\ell = 0$ ($u \leq \ell \leq u'$). If E_ℓ is not disjointly shared, then by $\omega.1$ E_ℓ is sensitive and since $j_1, j_2 \in f^*(E_u)$, this contradicts $\omega.2$. Thus E_ℓ is jointly shared. Then, using $\omega.6$ and $\phi.3$, the subsequences of (11.3) and (11.4) containing the bonsais $E_u, E_{u+1}, \dots, E_{u'}$ are justified.

Suppose that there exists at least one sensitive bonsai $E_\ell \notin U$ such that $j_1 \in f^*(E_\ell)$ (the case $j_2 \in f^*(E_\ell)$ is similar). By $\phi.2$, E_ℓ is unique. Since $j_1 \in f^*(E_\ell) \cap f^*(E_u)$ for any $E_u \in U$, by Claim 0, it follows that $f_{S_2}(E_\ell) \cap f_{S_2}(E_u) = \emptyset$ for all $E_u \in U$. Therefore, by $\omega.3$ (in case $|U| = 1$) or $\omega.4$, $E_{u'} \prec^{j_1} E_\ell$. Finally, using $\omega.8$ or $\phi.4$ for dealing with S_1 or S_2 -dominated bonsais, this completes the proof. \blacksquare

Claim 3. Let E_ℓ be a sensitive bonsai such that $j_1, j_2 \notin f^*(E_\ell)$. If $u \neq u'$, or $u = u'$ and $I(E_u) \neq \emptyset$, then exactly one of the following cases holds.

- (i) For all $j \in f_{S_2}(E_\ell)$, we have $E_{uj} = I(E_u)$ if $u = u'$, or $E_{uj} = I_1(E_u)$ otherwise.
- (ii) There exists an index $i(\ell)$ such that $u \leq i(\ell) < u'$, $E_{i(\ell)} \prec^{j_1} E_\ell$ and $E_{i(\ell)+1} \prec^{j_2} E_\ell$.
- (iii) For all $j \in f_{S_1}(E_\ell)$, we have $E_{uj} = I(E_u)$ if $u = u'$, or $E_{uj} = I_2(E_u)$ otherwise.

Proof of Claim 3. This follows from $\omega.3$, $\omega.4$, $\phi.5$ and Claim 2. ■

Claim 4. There exists a forest $T_1(j_1, j_2)$ in D with vertex set $X_1(j_1, j_2)$ such that for any $E_\ell, E_{\ell'} \in X_1(j_1, j_2)$ with $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$, E_ℓ and $E_{\ell'}$ are contained in a same subpath of $T_1(j_1, j_2)$.

Proof of Claim 4. We first make two observations:

- 1) For any pair of bonsais E_ℓ and $E_{\ell'}$ in $X_1(j_1, j_2)$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$, we have $(E_\ell, E_{\ell'}) \in D$ or $(E_{\ell'}, E_\ell) \in D$.
- 2) For any $E_\ell, E_{\ell'} \in D$, we have $(E_\ell, E_{\ell'}) \in D \Rightarrow f^*(E_\ell) \subseteq f^*(E_{\ell'})$.

The first observation follows from the right-feasibility of any pair of bonsais in $X_1(j_1, j_2)$, while the second results from the definition of D . Let us prove that for any $W \subseteq X_1(j_1, j_2)$, W has the property claimed for $X_1(j_1, j_2)$. One proceeds by induction on the cardinality of the subsets of $X_1(j_1, j_2)$.

Let $W \subseteq X_1(j_1, j_2)$. If $|W| = 1$, then the proof is clear. Suppose now that $|W| \geq 1$, W is a proper subset of $X_1(j_1, j_2)$ and there exists a forest G with vertex set W satisfying the desired property. Let $E_\ell \in X_1(j_1, j_2) \setminus W$. Consider the set $W' = \{E_{\ell'} \in W : f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset\}$. By the second observation, we deduce that W' is closed in G . Denote by E_{r_1}, \dots, E_{r_k} the root vertices of the trees in $G(W')$.

Suppose that $G(W')$ contains at least two trees ($k \geq 2$). By induction hypothesis $f^*(E_{r_1}) \cap f^*(E_{r_2}) = \emptyset$. On the other hand, by the first observation, $(E_\ell, E_{r_1}) \in D$ or $(E_{r_1}, E_\ell) \in D$. If $(E_\ell, E_{r_1}) \in D$, then by observation 2, $f^*(E_\ell) \subseteq f^*(E_{r_1})$ and since $f^*(E_\ell) \cap f^*(E_{r_2}) \neq \emptyset$, it follows that $f^*(E_{r_1}) \cap f^*(E_{r_2}) \neq \emptyset$, a contradiction. We deduce that $(E_{r_i}, E_\ell) \in D$ for $i = 1, \dots, k$. Adding these edges in $G(W')$ results in a forest of D with vertex set $W \cup \{E_\ell\}$ satisfying the claimed property.

Now suppose that $G(W')$ contains exactly one tree ($k = 1$). Using $G(W')$, we construct a forest spanning $W' \cup \{E_\ell\}$ and satisfying the claimed property. Let E_u be the bonsai in $G(W')$ of largest height such that $(E_\ell, E_u) \in D$. As in the previous case, we can show that if any two bonsais, say $E_{u'}$ and $E_{u''}$, in $G(W')$ are not in a same subpath of $G(W')$, then $(E_{u'}, E_\ell) \in D$ and $(E_{u''}, E_\ell) \in D$. This implies that in the tree $G(W')$ the successor of E_u (if it exists) has exactly one predecessor (E_u). If E_u exists, adding (E_ℓ, E_u) and $(E_{u'}, E_\ell)$ in $G(W')$ and removing $(E_{u'}, E_u)$, for each predecessor $E_{u'}$ of E_u in $G(W')$, results in a forest satisfying the claimed property. Otherwise, adding (E_{r_1}, E_ℓ) in $G(W')$ yields the desired forest. ■

Below, we describe a subroutine of CentralI, called GAR, which constructs a basic $\{1, \rho\}$ -central representation of $A'_{R(j_1, j_2)\bullet}$, where $A'_{\bullet \times \{1, \dots, m\}} = A$, provided that A is $\{\epsilon, \rho\}$ -central. In the following procedure, if we are dealing with a network representation of a matrix related to a bonsai E_ℓ (for instance L_ℓ , $L_{\ell,1}$, A_ℓ^\cap , or N_ℓ) such that $x_\ell = 0$ and $j_k \in f^*(E_\ell)$ for some $k \in \{1, 2\}$, then we denote by $v_{\ell,k}$ the endnode of the path with edge index set $E_{\ell j_k}$ incident with e_1 (for $k = 1$) and e_ρ (for $k = 2$), and by $w_{\ell,k}$ the other endnode of this path.

For every sensitive bonsai E_ℓ such that $j_1, j_2 \notin f^*(E_\ell)$, we define a vertex v_ℓ^* as follows. If $u = u'$ and $I(E_u) = \emptyset$, then $v_\ell^* = v_{u,1} = v_{u,2}$; otherwise, in case (i) (resp., (ii) and (iii)) of Claim 3, let $v_\ell^* = v_{u,1}$ (resp., $v_\ell^* = v_{i(\ell),2}$ and $v_\ell^* = v_{u',2}$).

Procedure GAR($A, \{\epsilon, \rho\}$)**Input:** A non-network matrix A and two row indexes ϵ and $\rho \neq 1$.**Output:** A basic $\{1, \rho\}$ -central representation $G(A'_{R(j_1, j_2) \bullet})$ for some row index subset $R(j_1, j_2)$ of A , where $j_1 \in S_1$, $j_2 \in S_2$, or determines that A is not $\{\epsilon, \rho\}$ -central.

- 1) call **Truthassignment**($A, \{\epsilon, \rho\}$) outputting U , (j_1, j_2) and a truth assignment of $\Omega(U, j_1, j_2)$, or the fact that A is not $\{\epsilon, \rho\}$ -central;
- 2) consider basic network representations $G(L_u)$ (in case $U = (E_u)$), $G(L_{u,1})$ and $G(L_{u',2})$ (in case $U = (E_u, E_{u'})$), $G(A_\ell^\cap)$ for any jointly shared bonsai $E_\ell \notin U$ such that $x_\ell = 0$, and $G(N_\ell)$ for every $E_\ell \in V(j_1, j_2) \setminus \{E_u, E_{u+1}, \dots, E_{u'}\}$; contract all artificial edges and create the adjacent edges $e_1 = [w_1, w_\rho]$ then $e_\rho = [w_{\rho-1}, w_\rho]$;
- 3) for $k = 1, 2$ and any two successive bonsais E_ℓ and $E_{\ell'}$ in the ordering (11.3) or (11.4) ($E_\ell \prec^{j_k} E_{\ell'}$) identify $v_{\ell',k}$ with $w_{\ell,k}$ and identify $v_{1,1}$ with w_1 and $v_{\ell_1+1,2}$ with $w_{\rho-1}$;
- 4) for every sensitive bonsai E_ℓ such that $j_1, j_2 \notin f^*(E_\ell)$, identify v_ℓ with v_ℓ^* ;
- 5) compute a forest $T_1(j_1, j_2)$ as in Claim 4;
 for every root vertex E_ℓ of $T_1(j_1, j_2)$, identify v_ℓ with w_ρ ;
 for any edge $(E_\ell, E_{\ell'})_{E_{\ell'}^k}$ in $T_1(j_1, j_2)$, identify v_ℓ with the endnode ($\neq v_{\ell'}$) of the $B_{\ell'}$ -path with edge index set $E_{\ell'}^k$ in $G(N_{\ell'})$;
 output the resulting bidirected graph $G(A'_{R(j_1, j_2) \bullet})$;

Proposition 11.24 *The output of GAR is correct.***Proof.** This results from Proposition 11.23 and Claims 0, 1, 2, 3 and 4. ■

Finally, assume that we are given a left-compatible set U , a U -spanning pair (j_1, j_2) such that the graph $F^*(\overline{V(j_1, j_2)})$ is bipartite and a truth assignment of $\Omega(U, j_1, j_2)$. Let $R_i(j_1, j_2) = \cup_{E_\ell \in X_i(j_1, j_2)} E_\ell$ for $i = 0$ and 1 . We state a last claim.

Claim 5. The graph $F^*(\overline{V(j_1, j_2)})$ is partitionable into two colour classes, say Y_0 and Y_1 , such that

- for any $E_\ell \in \overline{V(j_1, j_2)}$, if $E_\ell \in Y_0$ (respectively, $E_\ell \in Y_1$), then $Opposite(E_\ell) \subseteq R_1(j_1, j_2)$ (respectively, $Opposite(E_\ell) \subseteq R_0(j_1, j_2)$);
- for $k = 0$ and 1 , there exists a forest in D , denoted as T_k , with vertex set Y_k ; and for any E_ℓ and $E_{\ell'}$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$, either the bonsais are in two distinct colour classes, or they are contained in a same subpath of T_0 or T_1 .

Proof of Claim 5. Thanks to some clauses in $\Omega(U, j_1, j_2)$, up to a renumbering of the connected components of $F^*(\overline{V(j_1, j_2)})$, we may assume that for all $1 \leq \kappa \leq \xi$, $Opposite(\mathcal{U}_\kappa^0) \subseteq R_1(j_1, j_2)$ and $Opposite(\mathcal{U}_\kappa^1) \subseteq R_0(j_1, j_2)$. Let $Y_0 = \cup_{\kappa=1}^\xi \mathcal{U}_\kappa^0$ and $Y_1 = \cup_{\kappa=1}^\xi \mathcal{U}_\kappa^1$. Clearly, Y_0 and Y_1 yield a bipartition of $F^*(\overline{V(j_1, j_2)})$ into two colour classes. By definition of $F^*(\overline{V(j_1, j_2)})$, in each colour class, if two bonsais E_ℓ and $E_{\ell'}$ are such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$, then $(E_\ell, E_{\ell'}) \in D$ or $(E_{\ell'}, E_\ell) \in D$. Then, following the proof of claim 4, one proves the claim. ■

Suppose that the procedure GAR has output a basic $\{1, \rho\}$ -central representation $G(A'_{R(j_1, j_2) \bullet})$ of the matrix $A'_{R(j_1, j_2) \bullet}$. For any $k \in \{0, 1\}$ and every root vertex $E_\ell \in T_k$,

we define a node z_ℓ as follows. By claim 5, for all $j, j' \in f^*(E_\ell)$, $A'_{R(j_1, j_2) \times \{j\}} \cap R_k(j_1, j_2) = A'_{R(j_1, j_2) \times \{j'\}} \cap R_k(j_1, j_2)$. Let z_ℓ be the endnode, distinct from w_ρ , of the path in $G(A'_{R(j_1, j_2) \bullet})$ with edge index set $A'_{R(j_1, j_2) \times \{j\}} \cap R_k(j_1, j_2)$ for any $j \in f^*(E_\ell)$. We are now prepared to state the main procedure.

Procedure CentralI($A, \{\epsilon, \rho\}$)

Input: A non-network matrix A and two row indexes ϵ and $\rho \neq 1$ such that $S_0 = \emptyset$.

Output: A basic $\{\epsilon, \rho\}$ -central representation $G(A)$ of A , or determines that none exists.

- 1) call **GAR**($A, \{\epsilon, \rho\}$) outputting a basic $\{1, \rho\}$ -central representation $G(A'_{R(j_1, j_2) \bullet})$ where $R(j_1, j_2)$ is some row index subset of some matrix A' , or the fact that A is not $\{\epsilon, \rho\}$ -central;
- 2) for every $E_\ell \in \overline{V(j_1, j_2)}$, compute a v_ℓ -rooted network representation B_ℓ of N_ℓ , if one exists, otherwise STOP: output that A is not $\{\epsilon, \rho\}$ -central;
- 3) compute some forests T_0 and T_1 as in Claim 5;
- 4) for every root vertex E_ℓ of T_0 or T_1 , identify v_ℓ with z_ℓ ;
for any edge $(E_\ell, E_{\ell'})_{E_{\ell'}^k}$ in T_0 or T_1 , identify v_ℓ with the endnode ($\neq v_{\ell'}$) of the $B_{\ell'}$ -path with edge index set $E_{\ell'}^k$ in $B_{\ell'}$;
up to a renumbering of the basic edges e_1 and e_ϵ , output a basic $\{\epsilon, \rho\}$ -central representation of A ;

Proof of Theorems 11.1 and 11.3. Let us prove the correctness of the procedure CentralI. By Proposition 11.24, the subroutine GAR in step 1 outputs a basic $\{1, \rho\}$ -central representation $G(A'_{R(j_1, j_2) \bullet})$ for some row index subset $R(j_1, j_2)$ of A , where $j_1 \in S_1$, $j_2 \in S_2$, or determines that A is not $\{\epsilon, \rho\}$ -central. Then, in step 2, if one stops, then by Proposition 11.14, A is not $\{\epsilon, \rho\}$ -central; otherwise, using Claim 5, in step 4, a basic $\{1, \rho\}$ -central representation of A' is computed. At last, seeing step 1 of the subroutine Initialization, it is clear that the procedure CentralI outputs a basic $\{\epsilon, \rho\}$ -central representation of A if and only if one exists.

The proof of Theorem 11.3 follows from Theorem 11.1 and the tests and computations performed by the procedure CentralI.

Let us analyze the running time of CentralI. In the procedure Initialization, if E_ℓ is sensitive and $E_{\ell'}$ is S_1 , S_2 linked to E_ℓ in step 3, then the bonsai matrix associated with $E_\ell \cup E_{\ell'}$ is not a network matrix. Thus we essentially have to check whether the bonsai matrices N_1, \dots, N_b are network matrices, where E_1, \dots, E_b are given in step 2. By Theorem 2.5, this takes time $O(n\alpha)$. The computation of the digraph D described in Section 7.3 takes time $O(nm\alpha)$ by Lemma 7.11, and since $m \leq 4 \binom{n}{2} + 2n + 1$ (see the output of the procedure Camion) the required time is bounded by $Cn^3\alpha$ for some constant C .

Let U be a pair of shared bonsais as in step 4 of the procedure Truthassignment. By Theorem 11.33, if $U = E_u$, then computing a network representation of L_u in which e_1 and e_ρ are nonalternating takes time $O(n^2\alpha)$. One interesting observation is that for any bonsai E_ℓ , the matrices N_ℓ , A_ℓ , A_ℓ^\cap , L_ℓ , $L_{\ell,1}$ and $L_{\ell,2}$ associated with E_ℓ are independent from U . Thus, it is possible to compute network representations of these matrices, if they exist, before performing the computations in step 4 of the procedure Truthassignment. Checking whether every bonsai or pair of bonsais is left-feasible or right-feasible performs in time $O(n\alpha)$. Computing a truth assignment of $\Omega(U, j_1, j_2)$, for any U -spanning pair (j_1, j_2) , takes

time $O(n^2)$ which is bounded by $O(n\alpha)$, as $n^2 \leq n\alpha$. Finally, the computations in steps 2, 3 and 4 of the procedure CentralII can be executed in time $O(n\alpha)$. Altogether, the running time of CentralII is $O(n^3\alpha)$. ■

11.3 The procedure CentralII

Throughout this section, we assume that $S_0 \neq \emptyset$, and $\epsilon = 1$ (except in the procedures). We provide a proof of Theorems 11.2 and 11.4.

Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. A bonsai E_u ($1 \leq u \leq b$) is called *right-extreme* if B_u is S_0 -straight and on the right of $\{e_1, e_\rho\}$, and E_u has no S_0 -straight descendant in $T_{G_1(A)}$ (see page 98 for the definition of $T_{G_1(A)}$). The set of right-extreme bonsais is called the *right-extreme set*. Its cardinality is clearly at most two.

Like in the previous section, the notion of right-extreme set is related to that of right-compatibility defined as follows. We say that a set U of at most two S_0 -straight bonsais is *right-compatible* if the following holds.

- For any $E_u \in U$, $J_u^2 = \emptyset$ and N_u is a network matrix.
- There exists a right-feasible subgraph of D with vertex set $(V_0 \setminus V_{st}) \cup U$, and in the case where $U = \{E_u, E_{u'}\}$, E_u and $E_{u'}$ are not in a same subpath of this subgraph.

Lemma 11.25 *Suppose that A has a $\{1, \rho\}$ -central representation. Let U be the right-extreme set. Then U is right-compatible.*

Proof. By Lemma 11.8, for any $E_u \in U$, $J_u^2 = \emptyset$ and N_u is a network matrix. On the other hand, by Theorem 7.21 the forest $T_{G_1(A)}$ is a right-feasible subgraph of D . The vertex set $(V_0 \setminus V_{st}) \cup U$ might be not closed in $T_{G_1(A)}$. However, using the transitivity of the relation \prec_D , it results that there exists a right-feasible subgraph of D with vertex set $(V_0 \setminus V_{st}) \cup U$, and in the case where $U = \{E_u, E_{u'}\}$, E_u and $E_{u'}$ are not in a same subpath of this subgraph. (Otherwise, E_u (or $E_{u'}$) is a descendant of $E_{u'}$ (or E_u , respectively) in $T_{G_1(A)}$, contradicting the definition of a right-extreme set.) Therefore U is right-compatible. ■

The way of dealing with the graph $F^*(\overline{V'_0})$ is the same as with $F^*(\overline{V(j_1, j_2)})$ in Section 11.2. Let

$$R = \cup_{E_\ell \in V'_0} E_\ell \cup \{1, \rho\}.$$

Lemma 11.26 *If A has a $\{1, \rho\}$ -central representation $G(A)$, then the basic subgraph with edge index set R is a 1-tree.*

Proof. We know that the fundamental circuit of a nonbasic edge with index in S_0 contains the whole basic cycle, and for the rest of the proof see Lemma 11.18. ■

The Proposition 11.27 below shows the bipartiteness of the graph $F^*(\overline{V'_0})$, provided that A is $\{1, \rho\}$ -central. By assuming that $F^*(\overline{V'_0})$ is bipartite, we denote by $\mathcal{U}_1, \dots, \mathcal{U}_\xi$ the connected

components of $F^*(\overline{V'_0})$ and for each component \mathcal{U}_κ ($1 \leq \kappa \leq \xi$), let $\mathcal{U}_\kappa = \mathcal{U}_\kappa^0 \uplus \mathcal{U}_\kappa^1$ be a bipartition of \mathcal{U}_κ into two colour classes. For any $E_\ell \in \overline{V'_0}$, let

$$\text{Opposite}(E_\ell) = (\cup_{j \in f^*(E_\ell)} s(A_{\bullet j}) - \cap_{j \in f^*(E_\ell)} s(A_{\bullet j})) \cap R,$$

and for any $1 \leq \kappa \leq \xi$ and $i = 0$ and 1 , let $\text{Opposite}(\mathcal{U}_\kappa^i) = \cup_{E_\ell \in \mathcal{U}_\kappa^i} \text{Opposite}(E_\ell)$. The following proposition shows a part of Theorem 11.4.

Proposition 11.27 *Suppose that A has a basic $\{1, \rho\}$ -central representation $G(A)$. Then*

- (i) *The graph $F^*(\overline{V'_0})$ is bipartite.*
- (ii) *For any $1 \leq \kappa \leq \xi$, $i, i' \in \{0, 1\}$, $E_\ell \in \mathcal{U}_\kappa^i$ and $E_{\ell'} \in \mathcal{U}_\kappa^{i'}$, B_ℓ and $B_{\ell'}$ are at different sides of $\{e_1, e_\rho\}$ if and only if $i \neq i'$.*
- (iii) *For any $1 \leq \kappa \leq \xi$, $i \in \{0, 1\}$ and $E_\ell \in \mathcal{U}_\kappa^i$, the bonsai B_ℓ and the subgraph of $G(A)$ with edge index set $\text{Opposite}(\mathcal{U}_\kappa^i)$ are on both sides of $\{e_1, e_\rho\}$.*

Proof. The proposition can be proved along the same lines as Proposition 11.19. ■

Let U be a right-compatible set and $j_0 \in S_0$. We define relations \prec^1 and \prec^2 on $V_{st} \cup \{E_\ell : E_\ell \text{ is sensitive}\}$. For any $k \in \{1, 2\}$, $E_\ell \in V_{st}$ and $E_{\ell'} \in V_{st} \cup \{E_{\ell'} : E_{\ell'} \text{ is sensitive}\}$,

$$E_\ell \prec^k E_{\ell'} \Leftrightarrow f_{S_k}(E_{\ell'}) \subseteq \{\beta \in f_{S_k}(E_\ell) : E_{\ell\beta} = E_{\ell j_0}\}.$$

Clearly, the relations \prec^1 and \prec^2 are transitive. For all $1 \leq \ell \leq b$, let $E'_\ell = E_\ell \cup \{1, \rho\}$. A bonsai $E_\ell \in V'_0 \setminus U$ is said to be *right-feasible* if we have

$\mu.1$ $J_\ell^2 = \emptyset$ and N_ℓ is a network matrix;

$\mu.2$ if $E_\ell \in V_{st}$, then there exists some bonsai $E_u \in U$ such that $(E_\ell, E_u) \in D$ and for any $E_{\ell'} \in V_0 \setminus (V_2 \cup V_{st})$, if $(E_{\ell'}, E_u) \in D$ then $(E_{\ell'}, E_\ell) \in D$;

and *left-feasible* if we have

$\mu.3$ E_ℓ is sensitive or S_0 -straight;

$\mu.4$ if E_ℓ is S_0 -straight, then the matrix $L_\ell^0 = [A_{E'_\ell \times f(E_\ell)} \chi_{E_{\ell j_0} \cup \{1, \rho\}}^{E'_\ell}]$ is a network matrix;

Lemma 11.28 *Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$. Let U be the right-extreme set of bonsais. For any $E_\ell \in V'_0 \setminus U$, if the bonsai B_ℓ is on the right (respectively, the left) of $\{e_1, e_\rho\}$, then it is right-feasible (respectively, left-feasible).*

Proof. Let $E_\ell \in V'_0 \setminus U$. Suppose that B_ℓ is on the right of $\{e_1, e_\rho\}$. By Lemma 11.8, $J_\ell^2 = \emptyset$ and N_ℓ is a network matrix. Assume $E_\ell \in V_{st}$. By definition of U , E_ℓ has a descendant, say $E_u \in U$, in $T_{G_1(A)}$. Now let $E_{\ell'} \in V_0 - (V_{st} + V_2)$ such that $(E_{\ell'}, E_u) \in D$. If $f_{S_k}(E_{\ell'}) = \emptyset$ for all $k \in \{1, 2\}$, then clearly $(E_{\ell'}, E_\ell) \in D$. Otherwise, let $j \in f_{S_k}(E_{\ell'})$ for some $k \in \{1, 2\}$. Since $E_{\ell'} \notin V_{st} \cup V_2$, $(E_u, E_{\ell'}) \notin D$. As $j \in f_{S_k}(E_{\ell'}) \cap f_{S_k}(E_u)$, $E_{\ell'}$ is an ancestor of E_u in $T_{G_1(A)}$. Now, since E_ℓ is also an ancestor of E_u in $T_{G_1(A)}$ and $g_{j'}(E_\ell) = g_{j'}(E_{\ell'}) = g_{j'}(E_u) = 1$

for some $j' \in S_0$, it follows that E_ℓ , $E_{\ell'}$ and E_u are contained in a same subpath of $T_{G_1(A)}$. Finally, the fact that $E_{\ell'} \notin V_{st}$ and $E_\ell \in V_{st}$ implies that $(E_\ell, E_{\ell'}) \notin D$. So $(E_{\ell'}, E_\ell) \in D$.

Now suppose that B_ℓ is on the left of $\{e_1, e_\rho\}$. If $E_\ell \in V_0$, then by Lemma 11.6 (ii) B_ℓ contains at least one edge of the basic cycle, E_ℓ is S_0 -straight and $E_{\ell j} = I(v_{\ell,1}, v_{\ell,2})$ for any $j \in S_0$; we deduce that the matrix L_ℓ^0 is a network matrix. If $E_\ell \notin V_0$, then by definition of V'_0 the bonsai E_ℓ is shared, and it results from Lemmas 11.6 (ii) and 11.7 that E_ℓ is sensitive. ■

A pair of bonsais $E_\ell, E_{\ell'} \in V'_0 \setminus U$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$ is said to be *right-feasible* if we have

δ.1 if E_ℓ and $E_{\ell'}$ are S_k -linked for some $k \in \{1, 2\}$, then $(E_\ell, E_{\ell'}) \in D$ or $(E_{\ell'}, E_\ell) \in D$;

and *left-feasible* if the following holds.

δ.2 The bonsais are not both sensitive.

δ.3 If they are both S_0 -straight, then $E_\ell \prec^k E_{\ell'}$ and $E_{\ell'} \prec^{k'} E_\ell$ for some $k, k' \in \{1, 2\}$, $k \neq k'$.

δ.4 if E_ℓ is S_0 -straight and $E_{\ell'}$ sensitive, then $E_\ell \prec^k E_{\ell'}$ for some $k \in \{1, 2\}$, and E_ℓ and $E_{\ell'}$ are not S_1, S_2 linked.

Lemma 11.29 Suppose that A has a $\{1, \rho\}$ -central representation $G(A)$ and assumption \mathcal{A} is satisfied. Let U be the right-extreme set of bonsais and $E_\ell, E_{\ell'} \in V'_0 \setminus U$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$. If B_ℓ and $B_{\ell'}$ are both on the right (respectively, the left) of $\{e_1, e_\rho\}$, then the pair is right-feasible (respectively, left-feasible).

Proof. Suppose first that B_ℓ and $B_{\ell'}$ are on the right of $\{e_1, e_\rho\}$ and they are S_k -linked for some $k \in \{1, 2\}$. Then they belong to a same subpath of $T_{G_1(A)}$. Using the transitivity of \prec_D , we get that $(E_\ell, E_{\ell'}) \in D$ or $(E_{\ell'}, E_\ell) \in D$.

Now suppose that B_ℓ and $B_{\ell'}$ are on the left of $\{e_1, e_\rho\}$. They can not be both sensitive (otherwise from Lemmas 11.12 and 11.9, it follows that $v_{\ell,1} = v_{\ell,2}$ and $v_{\ell',1} = v_{\ell',2}$, hence $f^*(E_\ell) \cap f^*(E_{\ell'}) = \emptyset$). If E_ℓ and $E_{\ell'}$ are both S_0 -straight, then the proof that they are left-feasible is straightforward. At last, assume that E_ℓ is S_0 -straight and $E_{\ell'}$ sensitive. By Lemma 11.12, $B_{\ell'}$ is not S_1, S_2 linked to B_ℓ , so that $B_{\ell'}$ is either preceding B_ℓ in which case $E_\ell \prec^2 E_{\ell'}$, or succeeding B_ℓ and $E_\ell \prec^1 E_{\ell'}$. ■

Assume that the graph $F^*(\overline{V'_0})$ is bipartite. Given a right-compatible set U , we are now ready to construct the instance $\Lambda(U)$ of the 2-SAT problem. The set of variables is $\{x_\ell : E_\ell \in V'_0\}$. Provided that A is $\{1, \rho\}$ -central, the equalities $x_\ell = 0$ and $x_\ell = 1$ mean that the bonsai B_ℓ is on the left and on the right of $\{e_1, e_\rho\}$, respectively, in some $\{1, \rho\}$ -central representation of A .

For any $E_u \in U$, set $x_u = 1$ in $\Lambda(U)$. For any $E_\ell \in V'_0$, if E_ℓ is not right-feasible (respectively, left-feasible), then set $x_\ell = 0$ (respectively, $x_\ell = 1$). For any pair of bonsais $E_\ell, E_{\ell'} \in V'_0 \setminus U$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$, if the pair is not right-feasible (resp., left-feasible), then put the clause $\bar{x}_\ell \vee \bar{x}_{\ell'}$ (resp., $x_\ell \vee x_{\ell'}$) in $\Lambda(U)$. Thus if the pair is not right-feasible (respectively, left-feasible), then at most one of the variables x_ℓ and $x_{\ell'}$ has value 1 (respectively, 0).

For $\kappa = 1, \dots, \xi$, do as follows. For any $i \in \{0, 1\}$ and two variables x_ℓ and $x_{\ell'}$ such that $E_\ell \cap \text{Opposite}(\mathcal{U}_\kappa^i) \neq \emptyset$ and $E_{\ell'} \cap \text{Opposite}(\mathcal{U}_\kappa^i) \neq \emptyset$, put the equality $x_\ell = x_{\ell'}$ in $\Lambda(U)$. Moreover, if $\text{Opposite}(\mathcal{U}_\kappa^0) \neq \emptyset$ and $\text{Opposite}(\mathcal{U}_\kappa^1) \neq \emptyset$, choose some x_ℓ and $x_{\ell'}$ such that $E_\ell \cap \text{Opposite}(\mathcal{U}_\kappa^0) \neq \emptyset$ and $E_{\ell'} \cap \text{Opposite}(\mathcal{U}_\kappa^1) \neq \emptyset$. Then put the clauses $x_\ell \vee x_{\ell'}$ and $\bar{x}_\ell \vee \bar{x}_{\ell'}$ in $\Lambda(U)$. These clauses ensure that the variables x_ℓ and $x_{\ell'}$ have different values.

Proposition 11.30 *Suppose that A has a $\{1, \rho\}$ -central representation and assumption \mathcal{A} is satisfied. Then, there exists a right-compatible set U such that the instance $\Lambda(U)$ has a truth assignment.*

Proof. Let U be the right-extreme set. By Lemma 11.25, U is right-compatible. For each bonsai E_ℓ , let $x_\ell = 0$ if E_ℓ is on the left of $\{e_1, e_\rho\}$, and $x_\ell = 1$ otherwise. By Lemmas 11.28 and 11.29 and Proposition 11.27 (ii) and (iii), we deduce that this is a truth assignment. ■

The following procedure computes a truth assignment of $\Lambda(U)$ for some right-compatible set U , provided that A is $\{1, \rho\}$ -central.

Procedure Truthassignment($A, \{\epsilon, \rho\}$)

Input: A non-network matrix A and two row indexes ϵ and $\rho \neq 1$ such that $S_0 \neq \emptyset$.

Output: Either a right-compatible set U with a truth assignment of $\Lambda(U)$,
or determines that A is not $\{\epsilon, \rho\}$ -central.

- 1) call **Initialization**($A, \{\epsilon, \rho\}$) outputting a matrix A' ;
- 2) check whether $F^*(\bar{V}_0')$ is bipartite; if it is not, then output that A is not $\{\epsilon, \rho\}$ -central;
- 3) compute a digraph D with respect to A' and the row index subset $R^* = \{1, \rho\}$ of A' .
- 4) **for** every set U of S_0 -straight bonsais, $|U| \leq 2$, **do**
 compute whether U is right-compatible; if it is not, then return to 4;
 compute the instance $\Lambda(U)$ and a truth assignment of $\Lambda(U)$, if one exists,
 output it with U and **STOP**, otherwise return to 4;
endfor
 output that A is not $\{\epsilon, \rho\}$ -central;

Proposition 11.31 *The output of the procedure Truthassignment is correct.*

Proof. By Lemma 11.13, the subroutine Initialization outputs a matrix A' of size $n \times m'$ whose $A'_{\bullet\{1, \dots, m\}} = A$ and such that A' is $\{1, \rho\}$ -central if and only if A is $\{\epsilon, \rho\}$ -central, and assumption \mathcal{A} is satisfied for A' .

Suppose that A is $\{\epsilon, \rho\}$ -central. Then, let $G(A')$ be a $\{1, \rho\}$ -central representation of A' . In step 2, by Proposition 11.27 (i), the graph $F^*(\bar{V}_0')$ is bipartite. Finally, using Proposition 11.30, the procedure Truthassignment outputs a truth assignment of $\Lambda(U)$. This concludes the proof. ■

Before proving Theorem 11.4, we see some properties ensuing from a truth assignment of $\Lambda(U)$. Let U be a right-compatible set. If $|U| = 2$, then we write $U = (E_u, E_{u'})$, otherwise $U = (E_u)$. We suppose that each variable x_ℓ corresponding to a bonsai $E_\ell \in V_0'$ has received a value 0 or 1, yielding a truth assignment of $\Lambda(U)$. Let

$$X_k = \{E_\ell \in V_0' : x_\ell = k\}$$

for $k = 0$ and 1 . We notice that if a variable x_ℓ is equal to 0 , then it is left-feasible and the corresponding bonsai E_ℓ is either sensitive or S_0 -straight. Up to a renumbering of the bonsais in X_0 , we may assume that $X_0 \cap V_{st} = \{E_1, E_2, \dots, E_t\}$. We now state some useful claims.

Claim 0. For any sensitive bonsai E_ℓ , $x_\ell = 0$ and there is no bonsai $E_{\ell'}$ that is S_1 , S_2 linked to E_ℓ and such that $x_{\ell'} = 0$.

Claim 1. For any $E_\ell \in V_0' \setminus V_{st}$, $x_\ell = 0$ if and only if E_ℓ is sensitive.

Claim 2. Up to a renumbering of the bonsais, there exist orderings, say

$$E_1 \prec^1 E_2 \prec^1 \dots E_{t-1} \prec^1 E_t \quad (11.5)$$

and

$$E_t \prec^2 E_{t-1} \prec^2 \dots E_2 \prec^2 E_1 \quad (11.6)$$

of the set $X_0 \cap V_{st}$.

Claim 3. For every sensitive bonsai E_ℓ , either there exists an index $i(\ell)$ with $1 \leq i(\ell) \leq t$ such that $E_{i(\ell)} \prec^1 E_\ell$ and $E_{i(\ell)+1} \prec^2 E_\ell$, or $E_1 \prec^2 E_\ell$.

Claim 4. There exists a right-feasible forest, say TX_1 , in D with vertex set X_1 .

The proof of Claim 0 directly follows from $\mu.1$, $\mu.3$, $\delta.2$ and $\delta.4$. The proof of Claim 1 follows from Claim 0 and $\mu.3$. The proof of Claim 2 (respectively, 3) is close to the proof of Claim 2 (respectively, 3) in Section 11.2, using $\delta.3$ (respectively, $\delta.4$). The proof of Claim 4 is a consequence of the definition of a right-compatible set, $\mu.2$ and $\delta.1$.

Below, we describe a subroutine of CentralII, called GAR, which constructs a basic $\{1, \rho\}$ -central representation of a matrix $A'_{R\bullet}$, where $A'_{\bullet \times \{1, \dots, m\}} = A$, provided that A is $\{\epsilon, \rho\}$ -central. In the following procedure, for any S_0 -straight bonsai E_ℓ such that $x_\ell = 0$, if the matrix L_u^0 has a basic network representation $G(L_u^0)$, we denote by $v_{\ell,1}$ (respectively, $v_{\ell,2}$) the initial (respectively, terminal) node of the path with edge index set $E_{\ell,j}$ for any $j \in S_0$. At last, for every sensitive bonsai E_ℓ , we define a vertex v_ℓ^* as follows. If there exists an index $i(\ell)$ as defined in Claim 3, then $v_\ell^* = v_{i(\ell),2}$, otherwise $v_\ell^* = v_{1,1}$.

Procedure GAR($A, \{\epsilon, \rho\}$)

Input: A non-network matrix A and two row indexes ϵ and $\rho \neq 1$.

Output: A basic $\{1, \rho\}$ -central representation $G(A'_{R\bullet})$ of a matrix $A'_{R\bullet}$, or determines that A is not $\{\epsilon, \rho\}$ -central.

- 1) call **Truthassignment**($A, \{\epsilon, \rho\}$) outputting U and a truth assignment of $\Lambda(U)$, or the fact that A is not $\{\epsilon, \rho\}$ -central;
- 2) consider basic network representations $G(L_\ell^0)$ for any $E_\ell \in V_{st} \cap X_0$, and $G(N_\ell)$ for any $E_\ell \in V_0' \setminus (V_{st} \cap X_0)$; contract all artificial edges and create the edges $e_1 = [w_1, w_\rho]$ and $e_\rho = [w_{\rho-1}, w_\rho]$;
- 3) for any $1 \leq i \leq t-1$, identify $v_{i,2}$ with $v_{i+1,1}$, and identify $v_{1,1}$ with w_1 and $v_{t,2}$ with $w_{\rho-1}$;

- 4) for every sensitive bonsai E_ℓ , identify v_ℓ with v_ℓ^* ;
- 5) compute a forest TX_1 as in Claim 4;
 for every root vertex E_ℓ of TX_1 , identify v_ℓ with w_ρ ;
 for any edge $(E_\ell, E_{\ell'})_{E_{\ell'}^k}$ in TX_1 , identify v_ℓ with the endnode ($\neq v_{\ell'}$)
 of the $B_{\ell'}$ -path with edge index set $E_{\ell'}^k$ in $G(N_{\ell'})$;
 output the resulting bidirected graph $G(A'_{R_\bullet})$;

Proposition 11.32 *The output of GAR is correct.*

Proof. This results from Proposition 11.31 and Claims 0, 1, 2, 3 and 4. ■

Finally, assume that the graph $F^*(\overline{V'_0})$ is bipartite and we are given a right-compatible set U and a truth assignment of $\Lambda(U)$. We define $R_i = \cup_{E_\ell \in V_i} E_\ell$ for $i = 0$ and 1. We state a last claim.

Claim 5. The graph $F^*(\overline{V'_0})$ is partitionable into two colour classes, say Y_0 and Y_1 , such that

- for any $E_\ell \in \overline{V'_0}$, if $E_\ell \in Y_0$ (respectively, $E_\ell \in Y_1$), then $Opposite(E_\ell) \subseteq R_1$ (respectively, $Opposite(E_\ell) \subseteq R_0$);
- for $k = 0$ and 1, there exists a forest in D , denoted as T_k , with vertex set Y_k ; and for any E_ℓ and $E_{\ell'}$ such that $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$, either the bonsais are in two distinct colour classes, or they are contained in a same subpath of T_0 or T_1 .

Proof of Claim 5. The proof is identical to the proof of Claim 5 in Section 11.2. ■

Suppose that the procedure GAR has output a basic $\{1, \rho\}$ -central representation $G(A'_{R_\bullet})$ of a matrix A'_{R_\bullet} . For any $k \in \{0, 1\}$ and every root vertex $E_\ell \in T_k$, we define a node z_ℓ as follows. By claim 5, for all $j, j' \in f^*(E_\ell)$, $A'_{R \times \{j\}} \cap R_k = A'_{R \times \{j'\}} \cap R_k$. Let z_ℓ be the endnode, distinct from w_ρ , of the path in $G(A'_{R_\bullet})$ with edge index set $A'_{R \times \{j\}} \cap R_k$ for any $j \in f^*(E_\ell)$. We are now prepared to state the main procedure.

Procedure CentralII($A, \{\epsilon, \rho\}$)

Input: A non-network matrix A and two row indexes ϵ and $\rho \neq 1$ such that $S_0 \neq \emptyset$.

Output: A basic $\{\epsilon, \rho\}$ -central representation $G(A)$ of A , or determines that none exists.

- 1) call **GAR**($A, \{\epsilon, \rho\}$) outputting a basic $\{1, \rho\}$ -central representation $G(A'_{R_\bullet})$ of a matrix A'_{R_\bullet} , or the fact that A is not $\{\epsilon, \rho\}$ -central.
- 2) for every $E_\ell \in \overline{V'_0}$, compute a v_ℓ -rooted network representation B_ℓ of N_ℓ ,
 if one exists, otherwise STOP: output that A is not $\{\epsilon, \rho\}$ -central;
- 3) compute some forests T_0 and T_1 as in Claim 5;
- 4) for every root vertex E_ℓ of T_0 or T_1 , identify v_ℓ with z_ℓ ;
 for any edge $(E_\ell, E_{\ell'})_{E_{\ell'}^k}$ in T_0 or T_1 , identify v_ℓ with the endnode ($\neq v_{\ell'}$) of the $B_{\ell'}$ -path
 with edge index set $E_{\ell'}^k$ in $B_{\ell'}$;
 up to a renumbering of the basic edges e_1 and e_ϵ , output a basic $\{\epsilon, \rho\}$ -central
 representation of A ;

Proof of Theorems 11.2 and 11.4. Let us prove the correctness of the procedure CentrallI. By Proposition 11.32, the subroutine GAR in step 1 outputs a basic $\{1, \rho\}$ -central representation $G(A'_{R_\bullet})$, or determines that A is not $\{\epsilon, \rho\}$ -central.

If one stops in step 2, then by Proposition 11.14 A is not $\{\epsilon, \rho\}$ -central; otherwise, using Claim 5, in step 4, a basic $\{1, \rho\}$ -central representation of A' is computed. At last, seeing step 1 of the subroutine Initialization, it is clear that the procedure CentrallI outputs a basic $\{\epsilon, \rho\}$ -central representation of A .

The proof of Theorem 11.4 follows from Theorem 11.2 and the tests and computations performed by the procedure CentrallI. The analysis of the running time of CentrallI is the same as for Centrall. Thus, the number of operations in CentrallI is $O(n^3\alpha)$. \blacksquare

11.4 Recognizing $\{1, \rho\}$ -noncorelated network matrices

Let A be a nonnegative connected network matrix A of size $n \times m$, α the number of nonzero entries of A and $\rho > 1$ a row index. Let $G(A)$ be a basic network representation of A . Our present goal is to recognize whether A is $\{1, \rho\}$ -noncorelated. We analyze an algorithm which takes a basic network representation $G(A)$ as input. If A is $\{1, \rho\}$ -noncorelated, it outputs a basic network representation $G'(A)$ such that e_1 and e_ρ are alternating in $G(A)$ if and only if they are nonalternating in $G'(A)$. We will prove the following.

Theorem 11.33 *Given the matrix A there exists an algorithm which either determines that A is $\{1, \rho\}$ -corelated, or provides basic network representations $G(A)$ and $G'(A)$ such that e_1 and e_ρ are alternating in $G(A)$ if and only if they are nonalternating in $G'(A)$. The running time of the algorithm is bounded by $C(n^2\alpha)$ for some constant C .*

Suppose first that each column of A contains at most two nonzero entries. We consider the undirected graph $L(A)$ defined as follows. The vertex set is the row index set of A and two row indexes i and i' are adjacent if and only if $s(A_{\bullet j}) = \{i, i'\}$ for some column index j . With these preliminaries, we can state the following theorem.

Theorem 11.34 *Suppose that each column of A has at most two nonzero entries. Then the matrix A is $\{1, \rho\}$ -noncorelated if and only if there exists a cutvertex in $L(A)$ separating 1 from ρ .*

Proof. Suppose that there is no cutvertex separating 1 from ρ in $L(A)$. Then either 1 and ρ are the endnodes of a cut-edge in $L(A)$ and A is clearly $\{1, \rho\}$ -corelated, or the vertices 1 and ρ are contained in a 2-connected subgraph of $L(A)$. So assume that 1 and ρ are in a 2-connected subgraph of $L(A)$. By Menger's theorem (see [17] for example), the vertices 1 and ρ belong to a cycle C of $L(A)$. This implies that the subgraph of $G(A)$ with edge index set $V(C)$ represents a star. So, if both paths in C between 1 and ρ have an even length, then e_1 and e_ρ are alternating in $G(A)$, otherwise nonalternating. Therefore A is $\{1, \rho\}$ -corelated.

Conversely, let i^* be a cutvertex in $L(A)$ separating 1 from ρ . Let us denote by w_1 and w_2 the endnodes of e_{i^*} in $G(A)$. Let L_2 be the connected component of $L(A) \setminus \{i^*\}$ containing the vertex ρ . Observe that the subgraph of $G(A)$ with edge index set $V(L_2)$ (and no isolated vertex), call it T_2 , is connected and contains exactly one endnode of e_{i^*} , say w_2 . Now consider the following procedure.

Procedure Move-T2($G(A), i^*$)**Input:** A basic network representation $G(A)$ and a cutvertex i^* separating 1 from ρ .**Output:** A basic network representation $G'(A)$ such that e_1 and e_ρ are alternating in $G(A)$ if and only if they are nonalternating in $G'(A)$.

- 1) make T_2 loose from e_{i^*} by making a copy of w_2 , say w'_2 , in T_2 ;
 - 2) reverse the orientation of all edges in T_2 and identify w'_2 with w_1 ;
- output the obtained basic network representation $G'(A)$ of A ;

Let $G'(A)$ be output by the procedure Move-T2. We observe that the minimal paths linking e_{i^*} and e_1 in $G(A)$ and $G'(A)$ are identical. Moreover, e_2 and e_{i^*} are alternating in $G(A)$ if and only if they are alternating in $G'(A)$. Then, by construction, e_1 and e_ρ are alternating in $G'(A)$ if and only if e_1 and e_ρ are nonalternating in $G(A)$. ■

From now on, we assume that A has a column with at least three nonzeros. Let e_{i^*} be the middle edge of a basic path of length 3 in $G(A)$. (Such a path clearly exists.) Let $R^* = \{i^*\}$, D be the digraph with respect to R^* as constructed in Section 7.3 and $S^* = \{j : i^* \in s(A_{\bullet j})\}$. If two basic edges e_i and $e_{i'}$ are at different sides of e_{i^*} in $G(A)$, then they can not belong to a same bonsai. So D has at least two vertices. Two bonsais $E_\ell, E_{\ell'} \in D$ are said to be *independent* if there exist network representations $G'(A)$ and $G''(A)$ of A such that B_ℓ and $B_{\ell'}$ are at the same side of e_{i^*} in $G'(A)$ and at different sides of e_{i^*} in $G''(A)$; otherwise they are *dependent*.

Lemma 11.35 *If for some $1 \leq \ell \leq b$ either $1 \in E_\ell$ and $\rho = i^*$, or $\rho \in E_\ell$ and $1 = i^*$, or $1, \rho \in E_\ell$, then the following holds. The matrix A is $\{1, \rho\}$ -noncorelated if and only if N_ℓ is $\{1, \rho\}$ -noncorelated.*

Proof. The proof is straightforward. ■

For the remaining part of the section, we will need the following assumption.

assumption \mathcal{B} : The matrix $A_{\overline{\{i\}} \times \overline{f(\{i\})}}$ is connected for $i = 1$ and ρ , $i^* \neq 1, \rho$ and 1 and ρ are not contained in a same bonsai of D .

Whenever the assumption \mathcal{B} is satisfied, we will assume that $1 \in E_1$, and $\rho \in E_2$. Let us see an auxiliary lemma then an important theorem.

Lemma 11.36 *Suppose that the assumption \mathcal{B} is satisfied. In any basic network representation of A , if e_{i^*} , e_1 and e_ρ are contained in a same path, then e_{i^*} is between e_1 and e_ρ .*

Proof. Suppose by contradiction that e_1 lies between e_{i^*} and e_ρ in some path contained in $G(A)$. (The proof is similar with e_1 and e_2 interchanged.) Then the matrix $A_{\overline{\{1\}} \times \overline{f(\{1\})}}$ is not connected, a contradiction. ■

Theorem 11.37 *Suppose that the assumption \mathcal{B} is satisfied. Then, A is $\{1, \rho\}$ -noncorelated if and only if N_1 is $\{1, i^*\}$ -noncorelated, or N_2 is $\{\rho, i^*\}$ -noncorelated, or E_1 and E_2 are independent.*

Proof. Suppose first that N_1 and N_2 are $\{1, i^*\}$ -corelated and $\{\rho, i^*\}$ -corelated, respectively, and E_1 and E_2 are dependent. Consider the case where e_1 and e_{i^*} as well as e_ρ and e_{i^*} are nonalternating in all network representations of N_1 and N_2 , respectively; and suppose that B_1 and B_2 are at the same side of e_{i^*} in all network representations of A . Let us denote by $e_{i^*} = (w_1, w_2)$ in $G(A)$. Up to a reversing of the orientation of all edges, we may assume that $v_1^* = v_2^* = w_2$ (see page 94 for the definition of v_1^* and v_2^*).

Observe that e_1 and e_{i^*} as well as e_ρ and e_{i^*} are nonalternating in $G(A)$. (If e_1 and e_{i^*} are alternating in $G(A)$, then by contracting all edges of $G(A)$ with index in $\{1, \dots, n\} \setminus (E_1 \cup \{i^*\})$ we obtain a basic network representation of N_1 such that e_1 and e_{i^*} are alternating, a contradiction.) From Lemma 11.36, it results that e_1 (respectively, e_ρ) does not lie on the basic path from v_2 to w_2 (respectively, from v_1 to w_2). Thus e_1 and e_ρ are alternating in $G(A)$. In the other cases, one can also prove with similar arguments that A is $\{1, \rho\}$ -corelated.

Conversely, suppose that N_1 is $\{1, i^*\}$ -noncorelated. Let T_1 be a basic network representation of N_1 such that e_1 and e_{i^*} are alternating in T_1 if and only if they are nonalternating in $G(A)$. Recall that v_1 denotes the cutvertex of e_{i^*} in T_1 as well as the closest node of B_1 in $G(A)$ to e_{i^*} . Let T'_1 be the tree obtained from T_1 by contracting e_{i^*} . Observe that the subgraphs of T'_1 and $B_1 \subseteq G(A)$ with edge index set $\cup_{j \in f^*(E_1)} s(A_{\bullet, j}) \cap E_1$ are isomorphic and rooted at v_1 . Moreover, for all $j \in f^*(E_1)$, $s(A_{\bullet, j}) \cap E_1$ is the edge index set of a directed path in T'_1 and $B_1 \subseteq G(A)$. Thus it is possible to replace B_1 in $G(A)$ by T'_1 in order to obtain a basic network representation $G'(A)$ such that e_1 and e_ρ are alternating in $G(A)$ if and only if they are nonalternating in $G'(A)$. Similarly, if N_2 is $\{\rho, i^*\}$ -noncorelated, one can prove that A is $\{1, \rho\}$ -noncorelated.

At last, suppose that N_1 is $\{1, i^*\}$ -corelated, N_2 is $\{\rho, i^*\}$ -corelated and E_1 and E_2 are independent. We may suppose that B_1 and B_2 are at different sides of e_{i^*} in $G(A)$. Consider the case where e_1 (respectively, e_ρ) and e_{i^*} are nonalternating in all network representations of N_1 (respectively, N_2). Let $G'(A)$ be a basic network representation of A such that B_1 and B_2 are at the same side of e_{i^*} in $G'(A)$. Using Lemma 11.36, it follows that e_1 and e_ρ are not on the basic path from v_2 to w_2 and from v_1 to w_2 , respectively, in $G(A)$ and $G'(A)$. So e_1 and e_ρ are nonalternating in $G(A)$, and e_1 and e_ρ are alternating in $G'(A)$. The other cases can be analyzed in a same way. ■

By Lemma 11.35 and Theorem 11.37, our initial problem can be reduced to determining whether the bonsais E_1 and E_2 are independent. Consider the undirected graph $F^*(V)$ defined at page 140. Denote by $\mathcal{W}_1, \dots, \mathcal{W}_c$ the connected components of $F^*(V)$. We will prove the following proposition.

Proposition 11.38 *The bonsais E_1 and E_2 are independent if and only if E_1 and E_2 are in different connected components of $F^*(V)$.*

Let us see some lemmas that will be used for the proof of Proposition 11.38.

Lemma 11.39 *Let $E_\ell, E_{\ell'} \in \mathcal{W}_{c_1}$ for some $1 \leq c_1 \leq c$, and $E_u \in V \setminus \mathcal{W}_{c_1}$ such that $(E_\ell, E_u) \in D$. Then $(E_{\ell'}, E_u) \in D$.*

Proof. Suppose first that $(E_\ell, E_{\ell'}) \in F^*(V)$. Let $j \in f^*(E_\ell) \cap f^*(E_{\ell'})$. Since $(E_\ell, E_u) \in D$, $j \in f^*(E_u)$. So $j \in f^*(E_{\ell'}) \cap f^*(E_u)$, and as $(E_{\ell'}, E_u) \notin F^*(V)$ it follows that $(E_{\ell'}, E_u) \in D$ or

$(E_u, E_{\ell'}) \in D$. If $(E_u, E_{\ell'}) \in D$, then by transitivity of the relation \prec_D we have $(E_{\ell}, E_{\ell'}) \in D$, which contradicts $(E_{\ell}, E_{\ell'}) \in F^*(V)$. Thus $(E_{\ell'}, E_u) \in D$. If $(E_{\ell}, E_{\ell'}) \notin F^*(V)$, since there is a path in $F^*(V)$ between E_{ℓ} and $E_{\ell'}$, we may conclude as before. ■

Lemma 11.40 *Let $E_{\ell}, E_{\ell'} \in \mathcal{W}_{c_1}$, $E_u, E_{u'} \in \mathcal{W}_{c_2}$ for some $1 \leq c_1, c_2 \leq c$, $c_1 \neq c_2$, such that $(E_{\ell}, E_u) \in D$ and $(E_{u'}, E_{\ell'}) \in D$. Then $|\mathcal{W}_{c_1}| = |\mathcal{W}_{c_2}| = 1$ and $E_{\ell} \sim_s E_u$.*

Proof. Since $(E_{\ell}, E_u) \in D$, by Lemma 11.39 $(E_{\ell'}, E_u) \in D$. Similarly, since $(E_{u'}, E_{\ell'}) \in D$, $(E_u, E_{\ell'}) \in D$. Thus $E_{\ell'} \sim_s E_u$. Suppose by contradiction that $|\mathcal{W}_{c_1}| \geq 2$. (The case $|\mathcal{W}_{c_2}| \geq 2$ is symmetric.) Let $E_{\ell''} \in \mathcal{W}_{c_1}$ such that $(E_{\ell'}, E_{\ell''}) \in F^*(V)$. As $E_{\ell'} \sim_s E_u$, using Lemma 7.16, it follows that $(E_u, E_{\ell''}) \in F^*(V)$, contradicting the fact that $E_u \notin \mathcal{W}_{c_1}$. This completes the proof. ■

W.l.o.g, we may assume $E_1 \in \mathcal{W}_1$. For any bonsai $E_u \in D$, let $Des_{G(A)}(E_u)$ be the set of all descendants of E_u in the forest $T_{G(A)}$. Let $U = \{E_u \in V \setminus \mathcal{W}_1 : (E_1, E_u) \in D \text{ and } Des_{G(A)}(E_u) \cap \mathcal{W}_1 = \emptyset\}$.

Lemma 11.41 *The set U is a closed set in $T_{G(A)}$.*

Proof. Using the transitivity of the relation \prec_D , the proof is done. ■

By Lemma 11.41, it follows that there exists a path $\mathcal{P}(U)$ in $T_{G(A)}$ whose vertex set is equal to $\cup_{E_u \in U} (s(A_{\bullet j}) \cap E_u) \cup \{i^*\}$ for any $j \in f^*(E_1)$. Denote by z_1 and z_2 the endnodes of $\mathcal{P}(U)$ and let $\Gamma = \{E_{\ell} : v_{\ell} = z_1 \text{ or } z_2\}$.

Lemma 11.42 *For any $E_{\gamma} \in \Gamma$, the following holds.*

- (i) $E_{\gamma} \in \mathcal{W}_1$, or
- (ii) $(E_{\gamma}, E_{\ell}) \in D$ for some $E_{\ell} \in \mathcal{W}_1$, or
- (iii) $f^*(E_{\gamma}) \cap f^*(E_{\ell}) = \emptyset$ for all $E_{\ell} \in \mathcal{W}_1$.

Proof. Let $E_{\gamma} \in \Gamma$. Suppose that $E_{\gamma} \notin \mathcal{W}_1$ and $f^*(E_{\gamma}) \cap f^*(E_{\ell}) \neq \emptyset$ for some $E_{\ell} \in \mathcal{W}_1$. Since $(E_{\gamma}, E_{\ell}) \notin F^*(V)$, it results that $(E_{\gamma}, E_{\ell}) \in D$ or $(E_{\ell}, E_{\gamma}) \in D$. Suppose by contradiction that $(E_{\ell}, E_{\gamma}) \in D$. By Lemma 11.39, $(E_1, E_{\gamma}) \in D$. Observe that $Des_{G(A)}(E_{\gamma}) \subseteq U$ by definition of Γ . So $Des_{G(A)}(E_{\gamma}) \cap \mathcal{W}_1 = \emptyset$, hence $E_{\gamma} \in U$, which is impossible since $\Gamma \cap U = \emptyset$. Therefore $(E_{\gamma}, E_{\ell}) \in D$. This concludes the proof. ■

Let $\Gamma_I = \{E_{\gamma} \in \Gamma : E_{\gamma} \in \mathcal{W}_1 \text{ or } (E_{\gamma}, E_{\ell}) \in D \text{ for some } E_{\ell} \in \mathcal{W}_1\}$ and $\Gamma_{II} = \{E_{\gamma} \in \Gamma : f^*(E_{\gamma}) \cap f^*(E_{\ell}) = \emptyset \text{ for all } E_{\ell} \in \mathcal{W}_1\}$.

Lemma 11.43 *For all $E_{\gamma} \in \Gamma_I$ and $E_u \in U$, we have $(E_{\gamma}, E_u) \in D$.*

Proof. This directly follows from Lemma 11.39 and the transitivity of the relation \prec_D . ■

Lemma 11.44 *For all $E_\gamma \in \Gamma_I$ and $E_{\gamma'} \in \Gamma_{II}$, $f^*(E_\gamma) \cap f^*(E_{\gamma'}) = \emptyset$.*

Proof. Suppose by contradiction that $f^*(E_\gamma) \cap f^*(E_{\gamma'}) \neq \emptyset$ for some $E_\gamma \in \Gamma_I$ and $E_{\gamma'} \in \Gamma_{II}$. If $E_\gamma \in \mathcal{W}_1$, then this contradicts the definition of Γ_{II} . So $E_\gamma \notin \mathcal{W}_1$ and by definition of Γ_I , $(E_\gamma, E_\ell) \in D$ for some $E_\ell \in \mathcal{W}_1$. This implies that $f^*(E_\gamma) \subseteq f^*(E_\ell)$ for some $E_\ell \in \mathcal{W}_1$, and since $f^*(E_\gamma) \cap f^*(E_{\gamma'}) \neq \emptyset$ it results that $f^*(E_{\gamma'}) \cap f^*(E_\ell) \neq \emptyset$, a contradiction as before. ■

Proof of Proposition 11.38. Suppose first that the bonsais E_1 and E_2 belong to a same connected component of $F^*(V)$. Let $E_\ell, E_{\ell'}$ be a pair of adjacent bonsais in $F^*(V)$. Since $f^*(E_\ell) \cap f^*(E_{\ell'}) \neq \emptyset$, if B_ℓ and $B_{\ell'}$ are at the same side of e_{i^*} in $G(A)$, then E_ℓ and $E_{\ell'}$ belong to a same path in $T_{G(A)}$, therefore by transitivity of \prec_D $(E_\ell, E_{\ell'}) \in D$ or $(E_{\ell'}, E_\ell) \in D$, contradicting the fact that $(E_\ell, E_{\ell'}) \in F^*(V)$. Thus B_ℓ and $B_{\ell'}$ are at different sides of e_{i^*} in $G(A)$. This implies that in any network representation of A , the bonsais B_1 and B_2 are at different sides of e_{i^*} if and only if any path in $F^*(V)$ joining them has an odd length. Thus E_1 and E_2 are dependent.

Now suppose that E_1 and E_2 are in different connected components of $F^*(V)$. Up to a renumbering of the connected components of $F^*(V)$, we may always assume that $E_1 \in \mathcal{W}_1$ and $E_2 \in \mathcal{W}_2$. Furthermore, up to a renumbering of the bonsais in D , we show that we may assume $E_2 \notin \Gamma_I$ and $\text{des}_{G(A)}(E_2) \cap \Gamma_I = \emptyset$. Before proving that, we state the following claim.

Claim. Either $(E_2, E_\ell) \notin D$ for all $E_\ell \in \mathcal{W}_1$, or $(E_1, E_u) \notin D$ for all $E_u \in \mathcal{W}_2$, or $E_1 \sim_s E_2$.

Proof of Claim. Suppose that $(E_2, E_\ell) \in D$ for some $E_\ell \in \mathcal{W}_1$ and $(E_1, E_u) \in D$ for some $E_u \in \mathcal{W}_2$. Then, by Lemma 11.40, $E_1 \sim_s E_2$. ■

If $E_1 \sim_s E_2$, then up to a renumbering of the bonsais we may assume that E_1 is not a descendant of E_2 in $T_{G(A)}$ ($E_1 \notin \text{Des}_{G(A)}(E_2)$); therefore $(E_1, E_2) \in D$ and $\text{Des}_{G(A)}(E_2) \cap \mathcal{W}_1 = \emptyset$ (otherwise by Lemma 11.40 (with $l = 1$ and $u' = u = 2$) we get a contradiction), and so $E_2 \in U$. Hence $E_2 \notin \Gamma_I$ and $\text{des}_{G(A)}(E_2) \cap \Gamma_I = \emptyset$. Now if $E_1 \not\sim_s E_2$, then by interchanging the label of E_1 and E_2 if necessary and using the Claim above we may assume that $(E_2, E_\ell) \notin D$ for all $E_\ell \in \mathcal{W}_1$, hence $E_2 \notin \Gamma_I$ and $\text{des}_{G(A)}(E_2) \cap \Gamma_I = \emptyset$. Consider the following procedure.

Procedure Intervert- $\Gamma_I(G(A), E_1, E_2)$

Input: A basic network representation $G(A)$ and two bonsais E_1 and E_2 being in different connected components of $F^*(V)$ and such that $E_2 \notin \Gamma_I$ and $\text{des}_{G(A)}(E_2) \cap \Gamma_I = \emptyset$.

Output: A basic network representation $G'(A)$ such that B_1 and B_2 are at different sides of e_{i^*} in $G(A)$ if and only if they are at the same side of e_{i^*} in $G'(A)$.

- 1) for each $E_\ell \in \Gamma_I$, get the bonsai B_ℓ loose from z_1 or z_2 by making a copy \tilde{v}_ℓ of $v_\ell = z_1$ or z_2 ;
- 2) for each $E_\ell \in \Gamma_I$, if v_ℓ was equal to z_1 (resp., z_2), identify \tilde{v}_ℓ with z_2 (resp., z_1); output the obtained basic network representation $G'(A)$ of A ;

By Lemmas 11.43 and 11.44, since $E_1 \in \Gamma_I$ or $\text{des}_{G(A)}(E_1) \cap \Gamma_I \neq \emptyset$, and $E_2 \notin \Gamma_I$ and $\text{des}_{G(A)}(E_2) \cap \Gamma_I = \emptyset$, B_1 is moved by the procedure $\text{Intervert-}\Gamma_I$, while B_2 is not moved. So the output of the procedure $\text{Intervert-}\Gamma_I$ is correct. \blacksquare

Proof of Theorem 11.33. We first construct a basic network representation $G(A)$. This is known to take time $O(n\alpha)$ (see Theorem 2.5). Since A is a network matrix, computing the digraph D with respect to some R^* ($|R^*| = 1$) takes time $O(n\alpha)$. So the following routines take time at most $C_0(n\alpha)$ for some constant C_0 :

- i) **if** each column of A has at most two nonzeros, **then**
 compute $L(A)$ and test if $L(A)$ has a cutvertex separating 1 from ρ ; if such a vertex say i^* exists, then call $\text{Move-T2}(G(A), i^*)$; otherwise output that A is $\{1, \rho\}$ -corelated;
endif
- (ii) **if** A has at least one column with three nonzeros, **then**
 choose a row index i^* such that $A_{\overline{\{i^*\}} \times \overline{f(\{i^*\})}}$ is not connected and compute the digraph D with respect to $R^* = \{i^*\}$;
 test if the assumption \mathcal{B} is satisfied; if it is, then test if E_1 and E_2 are in different connected components of $F^*(V)$; if they are, then relabel the bonsais or connected components of $F^*(V)$ so that $E_2 \notin \Gamma_I$ and $\text{des}_{G(A)}(E_2) \cap \Gamma_I = \emptyset$,
 call $\text{Intervert-}\Gamma_I(G(A), E_1, E_2)$;
endif

By Theorems 11.34 and 11.37, Lemma 11.35 and Proposition 11.38 (see also the proof of Proposition 11.38), if A stands each of these tests, then A is $\{1, \rho\}$ -noncorelated if and only if the matrix N_ℓ is $\{1, \rho\}$ -noncorelated for some $1 \leq \ell \leq b$ (in case where the assumption \mathcal{B} is not satisfied), or N_1 or N_2 is $\{1, i^*\}$ -noncorelated or $\{\rho, i^*\}$ -noncorelated, respectively. This is a recursive definition of a test for being $\{1, \rho\}$ -noncorelated. It results an algorithm which determines that A is $\{1, \rho\}$ -corelated, or provides a basic network representation $G'(A)$ such that e_1 and e_ρ are alternating in $G(A)$ if and only if they are nonalternating in $G'(A)$. We show that its running time is bounded by $Cn^2\alpha$ for some constant C , by induction on n . (The proof is inspired from [44].)

If the algorithm stops during the step i) or ii), then the time is bounded by $C_0n\alpha$ for some constant C_0 . Otherwise, in the worst case, we need to test the two bonsai matrices N_1 and N_2 for being $\{1, i^*\}$ -noncorelated and $\{\rho, i^*\}$ -noncorelated, respectively. This requires time at most $Cn_1^2\alpha_1 + Cn_2^2\alpha_2$ for some constant $C \geq C_0$, by the induction hypothesis, where r_ℓ (resp., α_ℓ) is the number of rows (resp., the number of nonzeros) of N_ℓ , for $l = 1$ and 2. Moreover, steps i) and ii) require time at most $C_0n\alpha$. Altogether, since $n_1 + n_2 = n + 1$, $2 \leq n_1, n_2$ and $\alpha_1, \alpha_2 \leq \alpha$, the time is bounded by:

$$C_0n\alpha + Cn_1^2\alpha_1 + Cn_2^2\alpha_2 \leq Cn^2\alpha,$$

for some constant $C \geq C_0$ and $n \geq 5$. This terminates the proof. \blacksquare

Chapter 12

Conclusion

In this last chapter we point out and discuss the main results and suggest some directions for further research.

The main goal of this work was to recognize binet matrices. To achieve this, given a matrix A , we developed a subroutine which computes a Camion basis of the matrix $[I \ A]$ in time bounded by a polynomial function of the size of A , or outputs that A is not binet. Then, we reduced the main problem to the recognition of nonnegative $\frac{1}{2}$ -binet matrices, nonnegative bicyclic matrices and nonnegative cyclic matrices. Some characterizations of these subclasses of matrices were provided.

Furthermore, we presented a new characterization of the Camion bases of the node-edge incidence matrix of any connected digraph, and a characterization of Camion bases of any given matrix with a polynomial-time recognition procedure. An algorithm which finds a Camion basis was also described.

Finally, for recognizing cyclic matrices, we solved as a subproblem the recognition of nonnegative $\{\epsilon, \rho\}$ -noncorelated network matrices, where ϵ and ρ are two given row indexes. We provided a nice characterization theorem for this class of matrices.

One can wonder whether the described method for recognizing binet matrices could be improved or whether there exists a simpler and more elegant method. Is it really necessary to find a Camion basis first for recognizing binet matrices? We may try to answer these questions.

The way of computing the digraph D in Chapter 7 should be improvable in terms of time efficiency. This would yield a faster algorithm for recognizing nonnegative R^* -cyclic matrices, nonnegative bicyclic matrices and $\frac{1}{2}$ -binet matrices. However, this does not modify the global complexity of the algorithm Binet. Moreover, the described method for recognizing nonnegative $\frac{1}{2}$ -binet matrices does not seem efficient since one computes several network representations of different submatrices. There should exist another faster method.

Two other directions for recognizing binet matrices have been proposed by Kotnyek [36]. The first one is to formulate the problem as a mixed integer programming problem, and then to solve it. This method does not lead to a polynomial-time algorithm so far. A second interesting approach is to convert first a given rational matrix of size $n \times m$ to an integral one by a finite number of steps; Kotnyek and Appa proved that even in the worst case we need only at most $2m$ pivoting operations to get an integral matrix, provided that the matrix we started with was binet. Then the goal is to find necessary and sufficient conditions for an integral matrix for being binet. The following theorem gives a necessary condition.

Theorem 12.1 (Kotnyek [36]) *If A is an integral binet matrix, then there exist network matrices N_1 and N_2 such that (a) $A = N_1 + N_2$, and (b) both $[N_1 \ N_2]$ and $\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ are network matrices.*

As a corollary, another necessary condition can be derived, similar to the well-known characterization of totally unimodular matrices due to Ghouila-Houri [29]. This latter claims that for each collection of columns of a totally unimodular matrix, there exists a scaling of the selected columns by ± 1 such that the sum of the scaled columns is a vector of $0, \pm 1$ elements, and the same is true for rows.

Theorem 12.2 (Kotnyek [36]) *For each collection of columns or rows of an integral binet matrix, there exists a scaling of the selected columns or rows by ± 1 such that the sum of the scaled columns or rows is a vector of $0, \pm 1, \pm 2$ elements.*

Unfortunately, these conditions are not sufficient, as matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ shows. The advantage of transforming first a given matrix to an integral one follows from Lemma 4.12: An integral matrix is binet if and only if it is cyclic or $\frac{1}{2}$ -binet. From my point of view, the main difficulty in this approach consists in recognizing cyclic $\{0, \pm 1\}$ -matrices. The natural method we use for recognizing $\{\epsilon, \rho\}$ -central $\{0, 1\}$ -matrices can not be directly adapted to $\{\epsilon, \rho\}$ -central $\{0, \pm 1\}$ -matrices for at least one reason: Given an $\{\epsilon, \rho\}$ -central $\{0, \pm 1\}$ -matrix A and an $\{\epsilon, \rho\}$ -central representation $G(A)$ of A , it happens that a fundamental circuit contains e_ϵ and e_ρ , but not the whole basic cycle in $G(A)$. On the contrary, in any $\{\epsilon, \rho\}$ -central representation $G(A)$ of a binet $\{0, 1\}$ -matrix A , if a fundamental circuit contains e_ϵ and e_ρ , then it contains the whole basic cycle in $G(A)$ (see Lemma 7.17). This small difference could be very hard to overcome.

A very nice and deep characterization of totally unimodular matrices over $GF(2)$ has been given by Tutte [54] and proved by Gerards [26] in a simple way. So, inspired by Theorem 4.31, we hope that there exists a simple characterization of integral binet matrices over $GF(3)$. Working on $GF(3)$ instead of \mathbb{R} , one can try to find a new algorithm for recognizing integral binet matrices.

An other possibility for further research is connected to matroids. How can we translate our results about binet matrices to statements about signed-graphic matroids? In particular, is it possible to determine whether a matroid is signed-graphic using our recognition procedure for binet matrices?

We would like to mention a last open problem: the recognition of 2-regular matrices. We believe that a combinatorial recognition algorithm should exist. Probably it would be similar to the recognition algorithm for totally unimodular matrices, originating from the decomposition theory of regular matroids, due to Seymour [46]. See also [42], for 2-sum and 3-sum of signed-graphic matroids.

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Languages

- French: native language
- English: fluent
- German & Italian: working knowledge

Procedure FOREST

July 22, 2008

If A is R -cellular, the following algorithm constructs a spanning forest T_F of D such that for all $\beta \in S$, the subgraph of T_F induced by $\{\Gamma_l \in V : \beta \in f(R^l)\}$, denoted by $T_F(\{\Gamma_l \in V : \beta \in f(R^l)\})$, is a β -fork, a β -path or a union of two disjoint β -paths. Moreover, if f_β is a non basic half-edge in some R -cellular representation of A , then $T_F(\{\Gamma_l \in V : \beta \in f(R^l)\})$ corresponds to a β -path.

If A has a R -cellular representation $G(A)$, the symbol T^* denotes the spanning forest of D defined in the previous section. We will assume that for each $\beta \in S$ and $1 \leq l \leq r'$, $s(A_{\bullet\beta}) \cap R^l$ has been partitioned into at most two subsets $\tilde{R}_{11}(\beta, l)$ and $\tilde{R}_{12}(\beta, l)$ and D has no closed directed path.

The construction of a forest T_F can be reduced to a sequence of 2-SAT problems. Assume that $T_{\bar{F}} = (U = \cup_{j=1}^{k-1} V_j, \bar{F})$ has been constructed and $V_k \subseteq V - U$. In the forest T^* , for all $\beta \in S$, $T^*(\{\Gamma_l \in V : \beta \in f(R^l)\})$ is a β -fork, a β -path or a union of two disjoint β -paths. So for all $\Gamma_u \in V$, if the tree $\Gamma_u \subseteq G(A)$ has a u -substem with edge index set equal to some R_h^u , then

$$\sum_{\Gamma_l : (\Gamma_l, \Gamma_u)_{R_h^u} \in T^*} \tilde{g}(\Gamma_l) \leq g(\Gamma_u). \quad (1)$$

The following definition is a quite natural consequence of this last property. An arc $(\Gamma_l, \Gamma_u)_{R_h^u}$ (with $\Gamma_l \in V_k$ and $\Gamma_u \in U$) is said to be *legal* if and only if $\tilde{g}(\Gamma_l) + \sum_{\Gamma_{u'} : (\Gamma_{u'}, \Gamma_u)_{R_h^u} \in \bar{F}} \tilde{g}(\Gamma_{u'}) \leq g(\Gamma_u)$.

We construct an instance I_k of 2-SAT. First we define the variables. For $\Gamma_l \in V_k$, if there exists a legal arc $(\Gamma_l, \Gamma_u) \in E$ labeled R_h^u , we define the variable $x_h^{(l,u)}$. The variable $x_h^{(l,u)}$ will be true if and only if we put (Γ_l, Γ_u) labeled R_h^u into \bar{F} . Moreover, if for all $\beta \in f(R^l)$, $T_{\bar{F}}(\{\Gamma_u \in U : \beta \in f(R^u)\})$ is empty, we define the variable x^l . And if for all $\beta \in f(R^l)$, $T_{\bar{F}}(\{\Gamma_u \in U : \beta \in f(R^u)\})$ is a β -path and $s_{\frac{1}{2}}(A_{\bullet\beta}) = \emptyset$, we also define x^l . The variable x^l has a true value if and only if we do not put any arc leaving Γ_l into \bar{F} meaning that Γ_l is a top vertex in T_F .

Now let see the clauses. For $\Gamma_l, \Gamma_{l'} \in V_k$ ($l \neq l'$), if there exist $\Gamma_u \in U$, $\beta \in f(R^l) \cap f(R^{l'})$ such that $(\Gamma_l, \Gamma_u)_{R_h^u}, (\Gamma_{l'}, \Gamma_u)_{R_h^u} \in E$ are legal and $\tilde{g}_\beta(\Gamma_l) + \tilde{g}_\beta(\Gamma_{l'}) + \sum_{\Gamma_{u'} : (\Gamma_{u'}, \Gamma_u)_{R_h^u} \in \bar{F}} \tilde{g}_\beta(\Gamma_{u'}) >$

$g_\beta(\Gamma_u)$, add into I_k the clause $x_h^{(l,u)} \vee x_h^{(l',u)}$. Thus if this clause is true, at most one of the arcs $(\Gamma_l, \Gamma_u)_{R_h^u}$ and $(\Gamma_{l'}, \Gamma_u)_{R_h^u}$ will be in T_F . And if x^l and $x^{l'}$ are well defined and $T_{\bar{F}}(\{\Gamma_u \in \bar{F} : \beta \in f(R^l)\})$ is a β -path for some $\beta \in f(R^l) \cap f(R^{l'})$, we include in I_k the clauses $x^l \vee x^{l'}$ and $x^l \vee x^{l'}$. So in a truth assignment, both clauses are satisfied if and only if

exactly one of the variables x^l or $x^{l'}$ is true. A variable of the type x^l or $x_h^{(l,u)}$ is said to be *associated* to the vertex $\Gamma_l \in D$.

If A has a R -cellular representation, we will prove that there are at most two variables associated to each vertex in V_k . If exactly one variable, say y , is associated to a vertex Γ_l , let $y = 1$ and if there are exactly two variables, say y_1 and y_2 , then we include in I_k the clauses $(y_1 \vee y_2)$ and $(\bar{y}_1 \vee \bar{y}_2)$. Note as before that if in a truth assignment we have $y_1 = y_2$, then only one of these two clauses is satisfied, whereas if $y_1 \neq y_2$ then both clauses are satisfied. Thus for all $\Gamma_l \in D$, exactly one variable associated to Γ_l has a true value.

Procedure FOREST

Input: A digraph $D = (V, E)$.

Output: A spanning forest $T_F = (V = U, F = \bar{F})$.

- 1) Let $\bar{F} = \emptyset$; $U = \emptyset$; $T_{\bar{F}} = (U, \bar{F})$; $\bar{D} = D$ and $k = 1$.
 - 2) **while** $\bar{D} \neq \emptyset$ **do**
 - 3) Build V_k the set of top vertices of \bar{D} .
 - 4) Find a truth assignment of I_k such that all clauses are satisfied if it exists, otherwise STOP. For each true variable related to an arc, add in \bar{F} the corresponding arc.
 - 5) $\bar{D} = \bar{D} - V_k$; $U = U \cup V_k$ and $k = k + 1$.
- endwhile**

Theorem 1 *Suppose A has a R -cellular representation $G(A)$. Then the algorithm FOREST does not stop at step 4) and for all $\beta \in S$, the subgraph $T_F(\{\Gamma_l \in V : \beta \in f(R^l)\})$ is a β -fork, a β -path or a union of two disjoint β -paths. Moreover, if f_β is a non basic half-edge in $G(A)$, then $T_F(\{\Gamma_l \in V : \beta \in f(R^l)\})$ corresponds to a β -path.*

Proof. Let $G(A)$ a R -cellular representation of A and T^* as defined in the previous section. For $\beta \in S$ and $U \subseteq V$, denote $U(\beta) = \{\Gamma_u \in U : \beta \in f(R^u)\}$. This is the set of vertices Γ_u in U such that f_β generates a u -substem in the tree Γ_u of $G(A)$. Let n the number of passages in loop 2 and for all $0 \leq k \leq n$, let $F_k := \bar{F}$ as in the algorithm FOREST after k passages in loop 2 and $T_k = (\cup_{j=1}^k V_j, F_k)$.

We will prove the following assertion by induction on k :

Affirmation: For all $\beta \in S$, the graph $T_k(\{\Gamma_l \in \cup_{j=1}^k V_j : \beta \in f(R^l)\})$ is a β -fork, a β -path or a union of two disjoint β -paths and the algorithm FOREST does not stop at step 4). Moreover, if f_β is a non basic half-edge in $G(A)$, then $T_k(\{\Gamma_l \in V : \beta \in f(R^l)\})$ corresponds to a β -path.

For $k = 0$, the affirmation is clearly true.

Let $k \geq 1$ and suppose that the affirmation is true for $k - 1$.

First prove that there are at most two variables associated to each vertex in V_k . Let $\Gamma_v \in V_k$. Clearly, the validity of the affirmation for $k - 1$ implies that there are at most two legal arcs leaving Γ_v . Moreover, if there are exactly two legal arcs leaving Γ_v , then for all $\beta \in f(R^v)$, $T_{k-1}(\{\Gamma_u \in U : \beta \in f(R^u)\})$ is a disjoint union of two β -paths or a β -fork. So x^v is not defined.

Let $U = \cup_{j=1}^{k-1} V_j$, $T_{k-1}^* = T^*(U)$ and $T_k^* = T^*(U \cup V_k)$. We will construct a set E_k of edges

such that E_k induces a truth assignment of I_k which satisfies all clauses with a true value. Let $\Gamma_{l_0} \in V_k$.

If there is an arc $(\Gamma_{l_0}, \Gamma_{u_0})$ in T_k^* labeled $R_h^{u_0}$, we distinguish two cases:

case a1) Each arc $(\Gamma_{l_0}, \Gamma_{u_0})$ is not legal.

case a2) Otherwise.

Now, if Γ_{l_0} has no successor in T_k^* , we still have two more cases:

case b1) $T_{k-1}(U(\beta))$ is a union of two disjoint β -paths having the root vertices Γ_{u_1} and $\Gamma_{u'_1}$ respectively, for some $\beta \in f(R^{l_0})$.

case b2) Otherwise.

Let make some remarks. Thanks to step 3) of the algorithm FOREST, we have:

For each arc $(\Gamma_l, \Gamma_{l'}) \in D$, if $\Gamma_l \in V_j$ and $\Gamma_{l'} \in V_{j'}$ for some j and j' , then $j > j'$. (2)

And for $\beta \in S$, if the set $\{\Gamma_u \in U : \tilde{g}_\beta(\Gamma_u) = 2\}$ is not empty, then $T_{k-1}(\{\Gamma_u \in U : \tilde{g}_\beta(\Gamma_u) = 2\})$ is a simple β -fork by the hypothesis of induction; using remark 2, we deduce:

$$T_{k-1}^*(\{\Gamma_l \in U : \tilde{g}_\beta(\Gamma_l) = 2\}) \approx T_{k-1}(\{\Gamma_l \in U : \tilde{g}_\beta(\Gamma_l) = 2\}). \quad (3)$$

In addition, we have:

For $\Gamma_l, \Gamma_{l'} \in V$, $\beta \in S$, if $\tilde{g}_\beta(\Gamma_l) = 1$ and $\tilde{g}_\beta(\Gamma_{l'}) = 2$, then $(\Gamma_l, \Gamma_{l'}) \in D$. (4)

The remark 4 follows from the existence of a path in T^* from Γ_l to $\Gamma_{l'}$ which is a simple β -fork and from the transitivity of the relation " $\Gamma_l \sim \Gamma_{l'} \Leftrightarrow (\Gamma_l, \Gamma_{l'}) \in E$ ".

Consider the case a1). Since $(\Gamma_{l_0}, \Gamma_{u_0})$ labeled $R_h^{u_0}$ is not legal, it follows:

$$\text{There exists } \delta \in f(R^l) \text{ such that } \tilde{g}_\delta(\Gamma_{l_0}) + \sum_{\Gamma_u : (\Gamma_u, \Gamma_{u_0})_{R_h^{u_0}} \in F_{k-1}} \tilde{g}_\delta(\Gamma_u) > g_\delta(\Gamma_{u_0}). \quad (5)$$

Assume $\tilde{g}_\delta(\Gamma_{l_0}) = 2$. From the remarks 2 and 4, we deduce that $U(\delta) = \{\Gamma_u \in U : \delta \in f(R^u)\} = \{\Gamma_l \in U : \tilde{g}_\delta(\Gamma_l) = 2\}$. From remark 3, it follows that $T_{k-1}^*(U(\delta)) \approx T_{k-1}(U(\delta))$. And since $(\Gamma_{l_0}, \Gamma_{u_0})_{R_h^{u_0}} \in T_k^*$, the vertex Γ_{u_0} is the root of the δ -path $T_{k-1}^*(U(\delta))$. So Γ_{u_0} has no predecessor in $T_{k-1}(U(\delta))$. Since $g_\delta(\Gamma_{u_0}) = 2$ (by definition $(\Gamma_{l_0}, \Gamma_{u_0}) \in E \Rightarrow \tilde{g}_\delta(\Gamma_{l_0}) \leq g_\delta(\Gamma_{u_0})$), the affirmation 5 can not be true. Thus $\tilde{g}_\delta(\Gamma_{l_0}) = 1$.

Assume $\tilde{g}_\delta(\Gamma_{u_0}) = 2$. See figure 1. The graph $T_{k-1}^*(U(\delta))$ is a simple δ -fork and $g_\delta(\Gamma_{u_0}) \geq 1$. Using remark 5, there exists $\Gamma_{u_1} \in U$ such that $(\Gamma_{u_1}, \Gamma_{u_0}) \in F_{k-1}$ and $\delta \in f(R^{u_1})$. If Γ_{u_0} has no predecessor in $T_{k-1}^*(U(\delta))$, then since $T_{k-1}^*(U(\delta))$ is a simple δ -fork, we deduce that $(\Gamma_{u_0}, \Gamma_{u_1}) \in E$ (and $(\Gamma_{u_1}, \Gamma_{u_0}) \in E$), which contradicts the assumption that there is no cycle in D . Let $\Gamma_{u'_0}$ the predecessor of Γ_{u_0} in $T_{k-1}^*(U(\delta))$ and $\Gamma_{u'_0}$ the root of the path $T_{k-1}^*(U(\delta))$. Remark that $(\Gamma_{u'_0}, \Gamma_u) \in E$ for all $\Gamma_u \in U(\delta)$ ($u \neq u'_0$). Since the set of neighbours of Γ_{l_0} in D

belongs to $U(\delta)$, we deduce that $(\Gamma_{l_0}, \Gamma_{u'_0}) \in E$ and $\Gamma_{u'_0} \in V_{k-1}$ (otherwise Γ_{l_0} would belong to some V_j with $j \leq k-1$). The vertex Γ_{u_0} has two predecessors in T^* (Γ_{l_0} and Γ_{v_δ}). So $T^*(U(\delta) \cup \{\Gamma_{l_0}\})$ is a non simple δ -fork. For all $\gamma \in f(R^{l_0})$, since $(\Gamma_{l_0}, \Gamma_{u'_0}) \in E$, we have that $\gamma \in f(R^{u'_0})$. Thus for all $\gamma \in f(R^{l_0})$, the graph $T^*(U(\delta) \cup \{\Gamma_{l_0}\})$ is a γ -fork. Now, since the γ -fork $T^*(U(\delta) \cup \{\Gamma_{l_0}\})$ is a closed set in T^* , if $U(\gamma) - U(\delta) \neq \emptyset$ for some $\gamma \in f(R^{l_0})$, then there exists $\Gamma_u \in U(\gamma) - U(\delta)$ such that (Γ_u, Γ_{l_0}) or $(\Gamma_u, \Gamma_{u'_0}) \in D$, which contradicts remark 2 (since $\Gamma_{l_0} \in V_k$ and $\Gamma_{u'_0} \in V_{k-1}$). Thus we have:

$$\text{For all } \gamma \in f(R^{l_0}), U(\gamma) = U(\delta) \text{ and } \tilde{g}_\gamma(\Gamma_{l_0}) = 1. \quad (6)$$

Suppose $T_{k-1}(U(\delta))$ is a simple δ -fork. In case $g_\delta(\Gamma_{u_0}) = 1$, by remark 1, the arc $(\Gamma_{v_\delta}, \Gamma_{u_0}) \in T_k^*$ has a label $R_{h'}^{u_0} \neq R_h^{u_0}$ and by transitivity of the relation " $\Gamma_l \sim \Gamma_{l'} \Leftrightarrow (\Gamma_l, \Gamma_{l'}) \in D$ ", there exists an arc $(\Gamma_{l_0}, \Gamma_{u_0})$ labeled $R_{h'}^{u_0}$. Then using remark 6, we can find a legal arc $(\Gamma_{l_0}, \Gamma_{u_0})$ with some label, contradicting the assumption of case a1). In case $g_\delta(\Gamma_{u_0}) = 2$, we may prove that $(\Gamma_{l_0}, \Gamma_{u_0})$ labeled $R_h^{u_0}$ is legal, which is a contradiction for the same reason. So $T_{k-1}(U(\delta))$ has two leaves $\Gamma_{u'_0}$ and an other say Γ_w . By remark 6, (Γ_{l_0}, Γ_w) is a legal arc with exactly one possible label since $\tilde{g}_\delta(\Gamma_w) = 1$.

Now assume $\tilde{g}_\delta(\Gamma_{u_0}) = 1$. See figure 2. Denote by $\Gamma_{v'_0}$ the root of the greatest δ -path or simple δ -fork in T_{k-1} containing Γ_{u_0} . By remark 5, we deduce that $\Gamma_{v'_0} \neq \Gamma_{u_0}$. Since there is no cycle in D , the path from Γ_{u_0} to a top vertex in T_{k-1}^* does not contain $\Gamma_{v'_0}$. So $T_{k-1}^*(U(\delta))$ is a non simple δ -forks or a disjoint union of two δ -paths whose leaves are Γ_{u_0} and say $\Gamma_{u'_0}$. Since Γ_{u_0} has a predecessor in $T_{k-1}(U(\delta))$, $\Gamma_{u_0} \notin V_{k-1}$. And as in the case " $\tilde{g}_\delta(\Gamma_{u_0}) = 2$ ", we deduce that $(\Gamma_{l_0}, \Gamma_{u'_0}) \in E$, $\Gamma_{u'_0} \in V_{k-1}$ and the remark 6 is also true. Let us remark that by remark 6, the arc $(\Gamma_{l_0}, \Gamma_{v'_0})$ is legal. If $\Gamma_{v'_0} = \Gamma_{u'_0}$, we may distinguish three cases:

- case $\alpha 1$) :** $T_{k-1}(U(\delta))$ has two leaves $\Gamma_{v'_0} = \Gamma_{u'_0}$ and say $\Gamma_{\bar{v}_1}$. The remark 6 implies that $(\Gamma_{l_0}, \Gamma_{\bar{v}_1})$ is a legal arc.
- case $\alpha 2$) :** $T_{k-1}(U(\delta))$ is a simple δ -fork. Let us call Γ_{u_c} the central node of $T_{k-1}(U(\delta))$. Since $(\Gamma_{l_0}, \Gamma_{u_0})$ and $(\Gamma_{l_0}, \Gamma_{u'_0}) \in E$, by remark 6 and the transitivity of the relation $\Gamma_l \sim \Gamma_{l'} \Leftrightarrow (\Gamma_l, \Gamma_{l'}) \in D$, $(\Gamma_{l_0}, \Gamma_{u_c})$ is a legal arc for some label.
- case $\alpha 3$) :** $T_{k-1}(U(\delta))$ is a δ -path.

In case b1), let $\delta \in f(R^{l_0})$ such that $T_{k-1}(U(\delta))$ is a union of two disjoint δ -paths with the roots Γ_{u_1} and $\Gamma_{u'_1}$ respectively. We have that $T_{k-1}^*(U(\delta))$ is a δ -path. We may suppose that $\Gamma_{u'_1}$ is the root of the path $T_{k-1}^*(U(\delta))$. As in the case a1), we may prove that $(\Gamma_{l_0}, \Gamma_{u'_1}) \in E$, $\Gamma_{u'_1} \in V_{k-1}$ and the remark 6 is also true. So $(\Gamma_{l_0}, \Gamma_{u_1})$ is a legal arc with exactly one possible label ($\tilde{g}_\delta(\Gamma_{u_1}) = 1$).

Consider the following procedure.

Procedure TruthAssignment

Input: The digraph $D = (V, E)$, T_{k-1} , T_k^* , U and V_k .

Output: E_k .

for each $\Gamma_{l_0} \in V_k$, **do**

We distinguish the following cases.

case a1): **if** $\tilde{g}_\delta(\Gamma_{u_0}) = 2$ **do**

Add into E_k the legal arc (Γ_{l_0}, Γ_w) .

otherwise ($\tilde{g}_\delta(\Gamma_{u_0}) = 1$)

If $\Gamma_{v'_0} \neq \Gamma_{u'_0}$, add into E_k the legal arc $(\Gamma_{l_0}, \Gamma_{v'_0})$.
 Otherwise, in case $\alpha 1$), add into E_k the legal arc $(\Gamma_{l_0}, \Gamma_{\bar{v}_1})$
 and in case $\alpha 2$), a legal arc $(\Gamma_{l_0}, \Gamma_{u_c})$ with some label.
endif
case a2): Add into E_k a legal arc $(\Gamma_{l_0}, \Gamma_{u_0})$ with label $R_h^{u_0}$ if possible.
case b1): Add into E_k the legal arc $(\Gamma_{l_0}, \Gamma_{u_1})$.
endfor

Now we have to show that for all $\beta \in S$, the subgraph of $(U \cup V_k, F_{k-1} \cup E_k)$ induced by $\{\Gamma_l \in U \cup V_k : \beta \in f(R^l)\}$, denoted by D_k^β , is a β -fork, a β -path or a union of two disjoint β -paths and if f_β is a half-edge, then D_k^β is a β -path. Thus E_k induces a truth assignment of I_k such that all clauses are true, which proves the affirmation for k .

If there are at least three vertices $\Gamma_{l_0}, \Gamma_{l_1}$ and Γ_{l_2} in V_k such that $\beta \in f(R^{l_i})$ for $i = 0, 1, 2$, then in the graph T^* , two of them are in a same β -path or simple β -fork and so adjacent in D by transitivity of the relation " $\Gamma_l \sim \Gamma_{l'} \Leftrightarrow (\Gamma_l, \Gamma_{l'}) \in D$ ", which is a contradiction, because $\{\Gamma_{l_0}, \Gamma_{l_1}, \Gamma_{l_2}\}$ represents a stable set in D .

Let $\Gamma_{l_0}, \Gamma_{l_1} \in V_k$ and suppose that there exists $\gamma \in f(R^{l_0}) \cap f(R^{l_1})$. Then D_k^γ is the subgraph of $(U \cup V_k, F_{k-1} \cup E_k)$ induced by $U(\gamma) \cup \{\Gamma_{l_0}, \Gamma_{l_1}\}$. Assume first that Γ_{l_0} satisfies case a1) ($(\Gamma_{l_0}, \Gamma_{u_0})_{R_h^{u_0}} \in T_k^*$). From remark 6 and since $T_k^*(U(\gamma) \cup \{\Gamma_{l_0}, \Gamma_{l_1}\})$ is a γ -fork or a disjoint union of two γ -paths, it follows that $(\Gamma_{l_1}, \Gamma_{u'_0}) \in T_k^*$. Since $\Gamma_{u'_0} \in V_{k-1}$, we deduce that the arc $(\Gamma_{l_1}, \Gamma_{u'_0})$ is a legal arc. So Γ_{l_1} satisfies the case a2). Then by the procedure TruthAssignment, $(\Gamma_{l_1}, \Gamma_{u'_0}) \in E_k$ and in all subcases of case a1), the graph D_k^γ is a γ -fork or a union of two disjoint γ -paths.

Now assume that Γ_{l_0} satisfies the case a2) and Γ_{l_1} does not satisfy the case a1). If Γ_{l_1} satisfies the case a2), then $(\Gamma_{l_0}, \Gamma_{u_0}), (\Gamma_{l_1}, \Gamma_{u_0}) \in T_k^*$ for some $\Gamma_{u_0} \in U$. So Γ_{l_0} is a central node of the γ -fork in T_k^* and for all $\Gamma_l \in U(\gamma)$, $\tilde{g}_\gamma(\Gamma_l) = 2$. By remark 3 and the procedure TruthAssignment, we deduce that D_k^γ is a γ -fork equal to $T_k^*(U(\gamma) \cup \{\Gamma_{l_0}, \Gamma_{l_1}\})$. If Γ_{l_1} has no successor in T_k^* , then we distinguish the two cases b1) and b2) for Γ_{l_1} . Let $\Gamma_{u'_1}$ the root of the path $T_{k-1}^*(U(\gamma))$. Since $(\Gamma_{l_0}, \Gamma_{u_0}) \in T^*$, u_0 must be equal to u'_1 . In case b1), let $\delta \in f(R^{l_1})$ such that $T_{k-1}(U(\delta))$ is a union of two disjoint δ -paths with the roots Γ_{u_1} and $\Gamma_{u'_1}$ respectively. By remark 6, $U(\delta) = U(\gamma)$. In case b2), $T_{k-1}(U(\gamma))$ is a γ -path. By the procedure TruthAssignment, $(\Gamma_{l_0}, \Gamma_{u_0})$ and $(\Gamma_{l_1}, \Gamma_{u_1}) \in D_k^\gamma$ in case b1) and $(\Gamma_{l_0}, \Gamma_{u_0}) \in D_k^\gamma$ in case b2). Thus D_k^γ is a union of two disjoint γ -paths.

At last, suppose that Γ_{l_0} and Γ_{l_1} satisfy the case b1) or b2). Then clearly D_k^γ is made up of two isolated nodes Γ_{l_0} and Γ_{l_1} . So D_k^γ is a union of two disjoint γ -paths.

If f_β is a half-edge for some $\beta \in S$ and $\beta \in f(R^l)$ for some $\Gamma_l \in V$, then by the hypothesis of induction $T_{k-1}(U(\beta))$ is a β -path. Let Γ_{u_0} the root of the path $T_{k-1}(U(\beta))$. We may prove that $T_{k-1}(U(\beta)) \approx T_{k-1}^*(U(\beta))$ and (Γ_l, Γ_{u_0}) is legal for some label. Thus by the procedure TruthAssignment, D_k^β is also a β -path.

We conclude that there exists a truth assignment of I_k such that all clauses are true and so the affirmation is true for k . ■

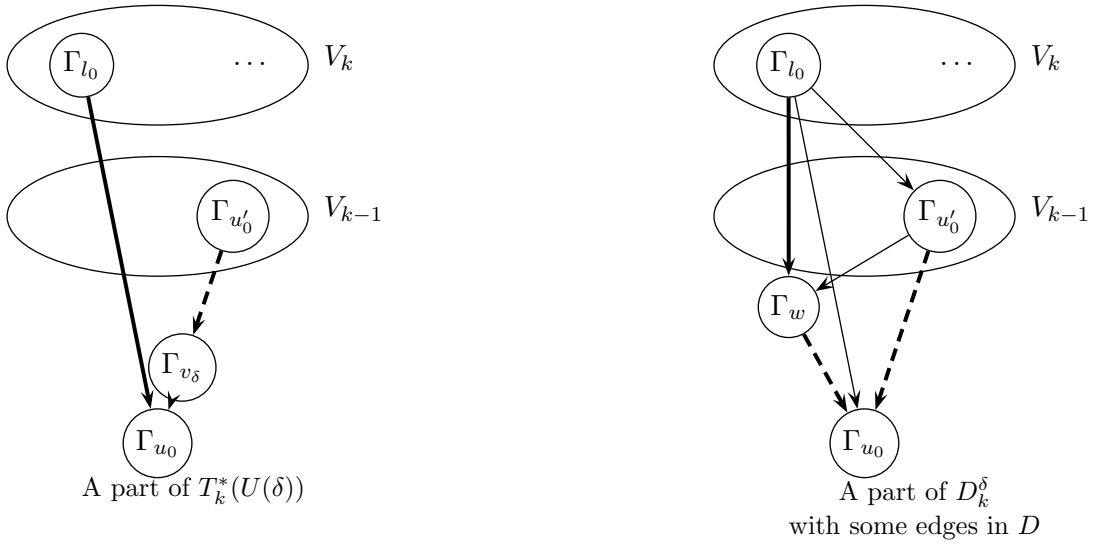


Figure 1: An illustration of case a1) with $\tilde{g}_\delta(\Gamma_{u_0}) = 2$. Heavy edges belong to $T_k^*(U(\delta))$ on the left and to D_k^δ on the right. A dashed edge corresponds to a path in $T_k^*(U(\delta))$ on the left and in D_k^δ on the right.

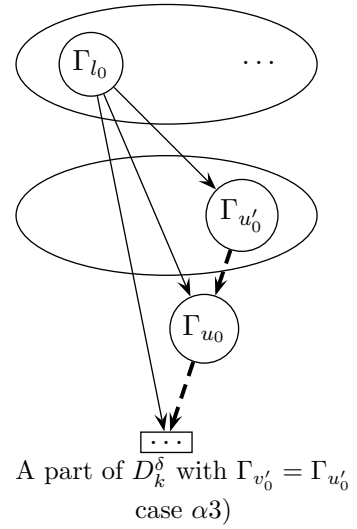
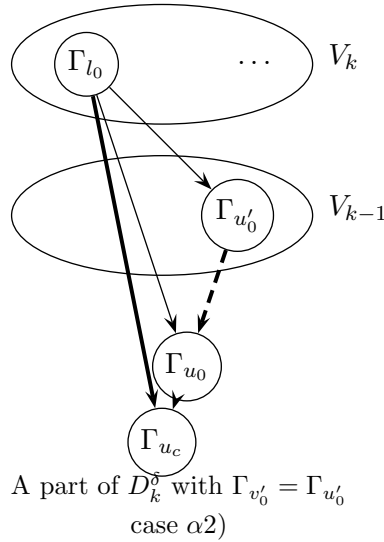
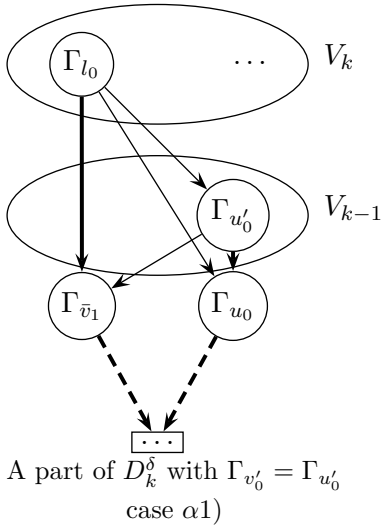
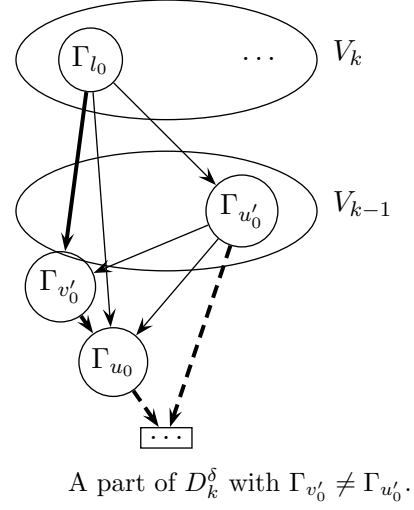
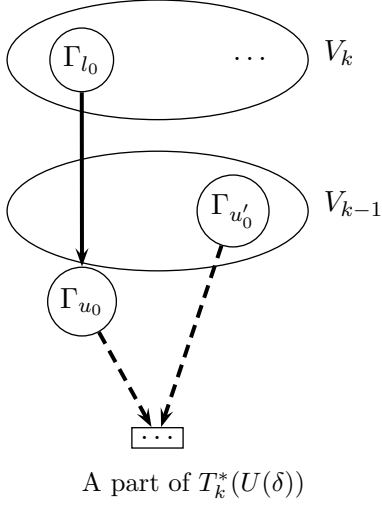


Figure 2: An illustration of case a1) with $\tilde{g}_\delta(\Gamma_{u_0}) = 1$. Heavy edges belong to $T_k^*(U(\delta))$ at the top left or D_k^δ otherwise. A dashed edge corresponds to a path in $T_k^*(U(\delta))$ at the top left or in D_k^δ otherwise. Thin edges belong to D .